Research Report

Macaulay dual generators of complete intersection ideals defined by complete homogeneous symmetric polynomials of successive degrees

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In this paper, we describe the contraction-annihilated Macaulay dual generators for complete intersection ideals defined by complete homogeneous symmetric polynomials of successive degrees. We also provide the first syzygy of the associated graded ring of each of these complete intersections with respect to the last variable. Using this, we prove that each of these complete intersections possesses the strong Lefschetz property, provided that the coefficient field of the polynomial ring has characteristic 0.

Keywords : Strong Lefschetz property, Macaulay dual generator, Artinian complete intersection algebra, Associated graded module, Complete homogeneous symmetric polynomial.

1. Introduction

Let $R=K[x_1,\cdots,x_n]$ be a polynomial ring in n variables over a field K, $\mathfrak{m}$ the graded maximal ideal of R, and I a $\mathfrak{m}$-primary homogeneous ideal of R with the standard grading, i.e., $\deg x_i=1$ for $i=1,\cdots,n$, and let $A=R/I$.

Definition 1.1. We say that $A=\bigoplus_{i=0}^{c}A_i$ has the *strong Lefschetz property* (SLP) if there exists a linear form $y\in A_1$ such that the multiplication map $\times y^d : A_i \to A_{i+d}$ has full rank for all $1\le d\le c$ and $0\le i\le c-d$.

The long-standing conjecture that every Artinian complete intersection should have the SLP has been studied by many authors but here we list only two survey papers (1) and (2) as references.

In this paper, we focus on a complete intersection ideal I defined by complete homogeneous symmetric polynomials of successive degrees, and let $A=R/I$.

In Section 2, we provide a first syzygy of the associated graded module $G^z(A)$ with respect to the last variable $z=x_n$. Using this, we give a proof that A has the SLP if K is a field of characteristic 0.

In Section 3, we present a contraction annihilated Macaulay dual generator of A and $G^z(A)$, which is effective for any characteristic of the coefficient field K, although the differential version is already known in characteristic 0 case (Example 2.87 in (1)).

2. Associated graded module with respect to the last variable

2.1 Preliminary

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Let $R' = K[x_1,\cdots,x_{n-1}] \subseteq R = K[x_1,\cdots,x_n] = R'[x_n]$ be a polynomial subalgebra of R in first $n-1$ variables. We often denote the last variable $z = x_n$. $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{\leq 0}$ denote the set of integers, non-negative integers, non-positive integers respectively. We sometimes use the multi-index notation for a monomial $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n} \in R$ and denote $\deg x^\alpha = |\alpha| = \alpha_1 + \cdots + \alpha_n$, where $\alpha = (\alpha_1,\cdots,\alpha_n) \in \mathbb{Z}_{\leq 0}^n$.

Notation 2.1.1. We assume that $n \geq 2$, where n is the number of variables.

(1) $h(a) = h_n(a) = \sum_{|\alpha|=a} x^\alpha \in R$ the complete homogeneous symmetric polynomial of degree a in n variables

for $a = 1,2,\cdots$, $h(0) = h_n(0) = 1$ and it is convenient to define $h(a) = h_n(a) = 0$ for $a < 0$.

(2) $h'(a) = h'_n(a) = \sum_{|\alpha'|=a} x^{\alpha'} \in R'$ the complete homogeneous symmetric polynomial of degree a in $n-1$ variables

for $a = 1,2,\cdots$, $h'(0) = h_{n-1}(0) = 1$ and $h'(a) = h_{n-1}(a) = 0$ for $a < 0$.

(3) $e_n(i) = (-1)^i \sum_{1\leq j_1<\cdots<j_i\leq n} x_{j_1}\cdots x_{j_i}$ the signed elementary symmetric polynomial of degree i in n variables for $i = 1,\cdots,n$,

$e_n(0) = 1$ and it is convenient to define $e_n(i) = 0$ for $i < 0$ or $i > n$.

(4) $\Psi = \prod_{i=0,\cdots,n-1} (z - x_i) = \sum_{i=0}^{n-1} e'(n-1-i) z^i \in R$, where $z = x_n$ and $e'(i) = e_{n-1}(i)$ the signed elementary symmetric polynomial of degree i in $n-1$ variables for $i = 0,\cdots,n-1$ and $e'(i) = 0$ for $i < 0$ or $i > n-1$.

By observing the product of the generation functions of complete homogeneous symmetric polynomials and signed elementary symmetric polynomials: $\left(\sum_{i=0}^{\infty} h_n(i) t^i\right)\cdot\left(\sum_{j=0}^{n} e_n(j) t^j\right) = \prod_{i=1}^{n} \frac{1}{1-x_i t} \cdot \prod_{i=1}^{n} (1 - x_i t) = 1$, we get the following well-known lemma.

Lemma 2.1.2. *The following holds:*

$$\sum_{j\in\mathbb{Z}} h_n(m-j) e_n(j) = \sum_{j=0}^{n} h_n(m-j) e_n(j) = \sum_{j=0}^{\min\{m,n\}} h_n(m-j) e_n(j) = 0 \quad \textit{for any integer } m \geq 1.$$

2.2 Associated graded module $G^z(R/I(a))$

Our main objects are ideals generated by complete homogeneous symmetric polynomials of successive degrees:

$$I(a) = (h(a), h(a+1), \cdots, h(a+n-1)) \subseteq R \quad \text{for } a \geq 1 \text{ and } n \geq 1 \text{ and}$$

$$I'(a) = (h'(a), h'(a+1), \cdots, h'(a+n-2)) \subseteq R' \quad \text{for } a \geq 1 \text{ and } n \geq 2.$$

For an ideal $I \subseteq R$ and $f \in R$, we denote $I : f = \{g \in R \mid gf \in I\}$. Maybe the following lemma is well-known, but we provide a proof for the sake of a self-contained explanation.

Lemma 2.2.1. *Let a be an integer with $a \geq 1$. Then the following holds:*

(1) $h(a+r) \in I(a)$ *for all integers* $r \geq 0$.

(2) $\sum_{j=0}^{n-1} h(a+n-1-j)e'(j) = z^{a+n-1}$ *and* $I(a) = (h(a),\cdots,h(a+n-2),z^{a+n-1})$.

(3) $I(a)$ *is a complete intersection ideal.*

(4) $I(a):z^j = (h(a),\cdots,h(a+n-2),z^{a+n-1-j})$ *for* $0 \le j \le n+a-1$.

(5) $$\frac{Rz+I(a):z^j}{Rz+I(a)} = \begin{cases} 0 & (j=1,\cdots,n+a-2) \\ R'/I'(a) & (j \ge a+n-1) \end{cases}.$$

Proof. (1) It is enough to show that $h(a+r) \in I(a)$ for $r \ge a+n-1$. We prover this by induction on $r \ge a+n-1$. If $r = a+n-1$, then clearly $h(a+n-1) \in I(a)$. Let $r > a+n-1$. By Lemma 2.1.2, we have $h_n(a+r) + \sum_{j=1}^{n} h_n(a+r-j)e_n(j) = 0$. Hence by the induction hypothesis and $a+r-n \ge a$, $h_n(a+r) = h(a+r) \in I(a)$.

(2) We only prove the first equation. Since $h(r) = \sum_{i=0}^{r} h'(r-i)z^i = \sum_{i\ge 0} h'(r-i)z^i$ for $r \ge 0$, we have

$$\sum_{j=0}^{n-1} h(a+n-1-j)e'(j) = \sum_{j=0}^{n-1}\sum_{i\ge 0} h'(a+n-1-j-i)z^i e'(j) = \sum_{i=0}^{a+n-1}\sum_{j=0}^{n-1} h'(a+n-1-i-j)e'(j)z^i$$

$$= z^{a+n-1} + \sum_{i=0}^{a+n-2}\sum_{j=0}^{n-1} h'(a+n-1-i-j)e'(j)z^i = z^{a+n-1}.$$

Here we remark that $\sum_{i=0}^{a+n-2}\sum_{j=0}^{n-1} h'(a+n-1-i-j)e'(j) = 0$ by Lemma 2.1.2. since $a+n-1-i \ge 1$ for $a+n-2 \ge i(\ge 0)$.

(3) Using (2), it is enough to show that $\dim_K \frac{R}{(h_n(a),\cdots,h_n(a+n-2),h_n(a+n-1))} = \dim_K \frac{R}{(h_n(a),\cdots,h_n(a+n-2),z^{a+n-1})} < \infty$.

We prove this by induction on n the number of variables. If $n=1$, then the assertion clearly holds. Let $n>1$. By the induction hypothesis, the following holds:

$$\dim_K \frac{R}{(h_n(a),\cdots,h_n(a+n-2),z)} = \dim_K \frac{R'}{(h_{n-1}(a),\cdots,h_{n-1}(a+n-2))} < \infty.$$

Hence we have $\dim_K \frac{R}{(h_n(a),\cdots,h_n(a+n-2),z^{a+n-1})} \le \dim_K \frac{R}{(h_n(a),\cdots,h_n(a+n-2),z)^{a+n-1}} < \infty$.

(4) By (3), $h(a),h(a+1),\cdots,z^{a+n-1}$ is a regular sequence in R. Therefore the assertion holds by (2).

(5) This follows from (4). □

$M(d)$ denotes the translation of a graded R-module M by degree $d \in \mathbb{Z}$, i.e., $M(d)_i = M_{d+i}$ for $i \in \mathbb{Z}$ where M_j denotes the degree $j \in \mathbb{Z}$ component of M. Let $I \subseteq R$ be a $\mathfrak{m}$-primary homogeneous ideal and put $A = R/I$ and $0 \ne y \in R_1$. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \to & \left(\frac{A}{0:y^{i+1}}\right)(-i-1) & \xrightarrow{\times y} & \left(\frac{A}{0:y^i}\right)(-i) & \to & \left(\frac{A}{yA+(0:y^i)}\right)(-i) & \to 0 \\ & \downarrow \varphi_1 = \times y^{i+1} & & \downarrow \varphi_2 = \times y^i & & \downarrow \varphi_3 = \times y^i & \\ 0 \to & y^{i+1}A & \to & y^i A & \to & \frac{y^i A}{y^{i+1}A} & \to 0 . \end{array}$$

Since φ_1 and φ_2 are isomorphisms, we have the following isomorphism: $\left(\dfrac{A}{yA+\left(0:y^i\right)}\right)(-i)\simeq\dfrac{y^iA}{y^{i+1}A}$.

Using this, we have also the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0\to & \left(\dfrac{yA+\left(0:y^{i+1}\right)}{yA+\left(0:y^i\right)}\right)(-i) & \to & \left(\dfrac{A}{yA+\left(0:y^i\right)}\right)(-i) & \to & \left(\dfrac{A}{yA+\left(0:y^{i+1}\right)}\right)(-i) & \to 0 \\
 & \downarrow \phi_1=\times y^i & & \downarrow \phi_2=\times y^i & & \downarrow \phi_3==\times y^{i+1} & \\
0\to & \ker\phi & \to & \dfrac{y^iA}{y^{i+1}A} & \overset{\times y=\phi}{\to} & \dfrac{y^{i+1}A}{y^{i+2}A}(1) & \to 0\ .
\end{array}$$

Since ϕ_2 and ϕ_3 are isomorphisms, we have the following remark.

Remark 2.2.2. The following holds for Artinian graded K-algebra $A=R/I$ and every non-negative integer i:

$$\ker\left(\frac{y^iA}{y^{i+1}A}\overset{\times y}{\to}\frac{y^{i+1}A}{y^{i+2}A}(1)\right)\simeq\frac{yA+\left(0:y^{i+1}\right)}{yA+\left(0:y^i\right)}\ \text{(up to shifting), where } 0\neq y\in R_1\ .$$

Definition 2.2.3. Let $A=R/I$ an Artinian graded K-algebra. We define the associated graded ring $G^y(A)$ with respect to $0\neq y\in R_1$ as follows: $G^y(A)=\bigoplus_{i\geq 0}G_i^y(A)$, where $G_i^y(A)=\dfrac{y^iA}{y^{i+1}A}$ for $i=0,1,\cdots$.

Notation 2.2.4. Let $I\subseteq R$ be a homogeneous ideal and $z=x_n$. $(I)_{z=0}$ is an ideal of R such that

$$(I)_{z=0}=\sum_{f\in I}\left(f|_{z=0}\right)R\text{, where } f|_{z=0} \text{ is a polynomial evaluated at } z=0\ .$$

Let t be an indeterminant. We can assume that $G^y(A)$ is a $G_0^y(A)[t]$-module by defining the t-action on $G^y(A)$ as follows:

$\begin{array}{ccc} G_i^y(A) & \overset{\times t}{\to} & G_{i+1}^y(A) \\ \| & \circlearrowright & \| \\ \dfrac{y^iA}{y^{i+1}A} & \overset{\times y}{\to} & \dfrac{y^{i+1}A}{y^{i+2}A}\end{array}$. Since $G^y(A)=\bigoplus_{i\geq 0}G_i^y(A)=\bigoplus_{i\geq 0}t^iG_0^y(A)$ is a principal $G_0^y(A)[t]$-module, we have

$\dfrac{G_0^y(A)[t]}{J}\simeq G^y(A)$ for some graded ideal $J\subseteq G_0^y(A)[t]$. Moreover, if $y=z=x_n$, then $G_0^z(A)=A/zA=R'/I'$ where $I'=(I)_{z=0}$. Therefore taking $t=z$, $G^z(A)\simeq\dfrac{(R'/I')[z]}{J}\simeq\dfrac{R'[z]}{\mathrm{in}^zI}$ for some homogeneous ideal $\mathrm{in}_zI\subseteq R'[z]=R$. Actually, $\mathrm{in}^zI=\sum_{i\geq 0}z^i\left(\mathrm{in}_i^zI\right)\subseteq R$, where $\mathrm{in}_i^zI=\left(I:z^i\right)_{z=0}$ for $i=0,1,\cdots$. We state this as the remark bellow.

Remark 2.2.5. $G^z(A)\simeq R/\mathrm{in}^zI$, where $\mathrm{in}^zI=\sum_{i\geq 0}z^i\left(\mathrm{in}_i^zI\right)\subseteq R$ with $\mathrm{in}_i^zI=\left(I:z^i\right)_{z=0}$ for $i=0,1,\cdots$.

We now recall the characterizations of the strong Lefschetz property (SLP) stated in [3] below.

Theorem 2.2.6. *Let $A=R/I$ an Artinian graded K-algebra and assume that the characteristic of K is 0. Then the following hold:*

(1) (Theorem 3.10 in (3)) $A\underset{K}{\otimes}K[t]/t^dK[t]$ *has the SLP for any integer* d *with* $d\geq 1$ *if* A *has the SLP.*

(2) (Theorem 4.6 in (3)) A *has the SLP if and only if* $G^y(A)$ *has the SLP for some* $0\neq y\in R_1$.

The following is our main theorem in this section.

Theorem 2.2.6. *Let* a,n *be integers with* $a\geq 1$ *and* $a\geq 2$*.Then the following hold:*

(1) $G^z(R/I(a))\simeq(R'/I'(a))\underset{K}{\otimes}\left(K[z]/z^{a+n-1}K[z]\right)\simeq R/\left(I'(a),z^{a+n-1}\right)$.

(2) $G^z(R/I(a))$ and $R/I(a)$ have the SLP if the characteristic of the coefficient field K is 0.

Proof. Let $A=R/I(a)$ then $G_0^z(A)\simeq R'/I'(a)$.

(1) By Lemma 2.2.1(5) and Remark 2.2.2, we have the following:

$$G^z(A)\simeq\bigoplus_{i=0}^{a+n-1}G_0^z(A)z^i\simeq G_0^z(A)\underset{K}{\otimes}\left(K[z]/z^{a+n-1}K[z]\right).$$

Or equivalently, using Remark 2.2.5, $\mathrm{in}^z I(a)=\left(I'(a),\,z^{a+n-1}\right)$ since by Lemma 2.2.1 (4), $I'R=\left(I:z^i\right)_{z=0}$ for $0\leq i<a+n-1$ and $R=\left(I:z^i\right)_{z=0}$ for $i\geq a+n-1$.

(2) We prove this by induction on $n\geq 2$ the number of variables.

Let $n=2$. Using (1), $G^z(A)\simeq G_0^z(A)\underset{K}{\otimes}\left(K[z]/z^{a+n-1}K[z]\right)$ has the SLP by Theorem 2.2.6(1), since both $G_0^z(A)\simeq K[x_1]/\left(x_1^a\right)$ and $K[z]/z^{a+1}K[z]$ have the SLP. Hence also A has the SLP by Theorem 2.2.6(2).

Let $n>2$. By the induction hypothesis, $G_0^z(A)\simeq R'/I'(a)$ has the SLP. Again using (1) and Theorem 2.2.6(1), $G^z(A)$ has the SLP. Hence also A has the SLP by Theorem 2.2.6(2). □

3. Macaulay dual

3.1 Contraction

Let $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{\leq 0}$ denote the set of integers, non-negative integers, non-positive integers respectively. We need several commutative K-algebras stated as below.

Notation 3.1.1.

(1) $R=K[x_1,\cdots,x_n]=R'[z=x_n]$ where $R'=K[x_1,\cdots,x_{n-1}]$.

(2) $R^\vee=K[x_1^{-1},\cdots,x_n^{-1}]$.

(3) $\mathcal{R}=K[x_1,\cdots,x_n,x_1^{-1},\cdots,x_n^{-1}]$ and $\mathcal{R}'=K[x_1,\cdots,x_n,x_1^{-1},\cdots,x_n^{-1}]$.

We introduce a partial order $\leq$ on $\mathbb{Z}^n$ as follows:

$$\alpha\leq\beta\overset{\text{def}}{\Leftrightarrow}\alpha_1\leq\beta_1,\cdots,\alpha_n\leq\beta_n \text{ for integer vectors } \alpha=(\alpha_1,\cdots,\alpha_n),\ \beta=(\beta_1,\cdots,\beta_n)\in\mathbb{Z}^n.$$

Notation 3.1.2.

$\left(\mathbb{Z}^n\right)_{\geq 0}=\left\{\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{Z}^n \mid \alpha_1\geq 0,\cdots,\alpha_n\geq 0\right\}$, $\left(\mathbb{Z}^n\right)_{\leq 0}=\left\{\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{Z}^n \mid \alpha_1\leq 0,\cdots,\alpha_n\leq 0\right\}$

We sometimes use the multi-index notation for monomials of Laurent monomial $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n} \in \mathcal{R}$, where $\alpha = (\alpha_1,\cdots,\alpha_n) \in \mathbb{Z}^n$. We need more notations before introducing the contraction.

Notation 3.1.3.

(1) $\pi : \mathcal{R} = K[x_1,\cdots,x_n,x_1^{-1},\cdots,x_n^{-1}] \to R^\vee = K[x_1^{-1},\cdots,x_n^{-1}]$ a K-linear map defined by $\pi(x^\alpha) = \begin{cases} x^\alpha & (\alpha \le 0) \\ 0 & (\text{others}) \end{cases}$.

(2) $(\ )^* : \mathcal{R} \to \mathcal{R}$ a K-algebra isomorphism defined by $(x^\alpha)^* = x^{-\alpha}$, which induces the K-algebra isomorphism between sub K-algebras R and $R^\vee$, i.e, induces a duality $(R)^* \simeq R^\vee$ and $(R^\vee)^* \simeq R$.

Definition 3.1.4.(Contraction) Let $f,g \in R \subseteq \mathcal{R}$, $F \in R^\vee \subseteq \mathcal{R}$ and $\xi,\zeta \in \mathcal{R}$.

(1) $\langle \_,\_ \rangle : \mathcal{R}\times\mathcal{R} \to \mathcal{R}$ K-bilinear map defined by $\langle \xi,\zeta \rangle = \pi(\xi\zeta^*)$.

(2) $\_\cdot\_ : R\times R^\vee \to R^\vee$ K-bilinear map defined by $f\cdot F = \pi(fF) = \langle f, F^* \rangle$.

(3) $\mathrm{cont}(\_,\_) : R\times R^\vee \to R^\vee$ K-bilinear map defined by $\mathrm{cont}(f,g) = \pi(f g^*) = \langle f,g \rangle$.

We call $\mathrm{cont}(f,g)$ "the contraction of g by f". Especially, $f\cdot F = \mathrm{cont}(f,F^*)$ and $\mathrm{cont}(f,g) = f\cdot g^*$.

Here we remark that the contraction operator is a variant of the differential operator. In this point of view, it is common to use the divided power algebra instead of the polynomial ring R but in this paper, we don't need the divided power structure.

Let $\mathrm{grMod}R$ denote the category of graded R-modules.

Remark 3.1.5. We remark that $R^\vee \in \mathrm{grMod}R$. Let $f,g \in R \subseteq \mathcal{R}$ and $F,G \in R^\vee \subseteq \mathcal{R}$. Actually, the following hold:

(1) $f\cdot(F+G) = \pi(f(F+G)) = \pi(fF+fG) = \pi(fF)+\pi(fG) = f\cdot F + f\cdot G$.

(2) $(f+g)\cdot F = \pi((f+g)\cdot F) = \pi(fF+gF) = \pi(fF)+\pi(gF) = f\cdot F + g\cdot F$.

(3) $(fg)\cdot F = f\cdot(g\cdot F)$ holds. Since for $\alpha,\beta \in (\mathbb{Z}^n)_{\ge 0}$ and $\gamma \in (\mathbb{Z}^n)_{\le 0}$, we have

$$x^\alpha\cdot(x^\beta\cdot x^\gamma) = x^\alpha\cdot\pi(x^{\beta+\gamma}) = \begin{cases} \pi(x^{\alpha+\beta+\gamma}) & (\beta+\gamma\le 0) \\ 0 & (\text{others}) \end{cases} = \begin{cases} x^{\alpha+\beta+\gamma} & (\alpha+\beta+\gamma\le 0) \\ 0 & (\text{others}) \end{cases} = (x^\alpha x^\beta)\cdot x^\gamma .$$

Here we use the fact that $\alpha+\beta+\gamma \le 0$ implies $\beta+\gamma\le 0$ ($\beta+\gamma \le \alpha+\beta+\gamma$) to prove the third equality of the above equation. For general $f = \sum_{\alpha\ge 0} c_\alpha x^\alpha$, $g = \sum_{\beta\ge 0} c_\beta x^\beta$, $F = \sum_{\gamma\le 0} c_\gamma x^\gamma$, using the K-linearity of this product,

$$f\cdot(g\cdot F) = \sum_{\alpha\ge 0,\beta\ge 0,\gamma\le 0} c_\alpha c_\beta c_\gamma x^\alpha\cdot(x^\beta\cdot x^\gamma) = \sum_{\alpha\ge 0,\beta\ge 0,\gamma\le 0} c_\alpha c_\beta c_\gamma (x^\alpha x^\beta)\cdot x^\gamma = (fg)\cdot F .$$

(4) $1\cdot F = \pi(F) = F$.

The following rules are convenient to calculate contractions by using the pairing $\langle \_,\_ \rangle$.

Remark 3.1.6. Let $f,g,h \in R$. The following hold:

(1) $\mathrm{cont}(f,gh) = \langle f,gh \rangle = \pi\left(f(gh)^*\right) = \pi(fg^*h^*) = \langle fg^*,h \rangle$.

(2) $\mathrm{cont}(fg,h)=\langle fg,h\rangle=\pi(fgh^*)=\pi(fg^*h^*)=\langle f,g^*h\rangle$.

(3) $\mathrm{cont}(f,g)=\langle fg^*,1\rangle=\langle 1,f^*g\rangle$.

The following lemma is quite simple but useful for calculating contractions.

Lemma 3.1.7. *Let* $\xi=\sum_{i\in\mathbb{Z}}c_i(\xi)z^i,\ \in\mathcal{R}$ *with* $c_i(\xi)\in\mathcal{R}'$ *for* $i\in\mathbb{Z}$ *and* $\zeta'\in\mathcal{R}'$. *Then the following holds:*

$$\langle\xi,\zeta'\rangle=\sum_{i\le 0}\langle c_i(\xi),\zeta'\rangle z^i=\langle\xi_{\le 0},\zeta'\rangle \quad \textit{where} \quad \xi_{\le 0}=\sum_{i\le 0}c_i(\xi)z^i .$$

Proof. $\langle\xi,\zeta'\rangle=\pi(\xi\zeta'^*)=\sum_{i\in\mathbb{Z}}\pi(c_i(\xi)\zeta'^*z^i)=\sum_{i\le 0}\pi(c_i(\xi)\zeta'^*)z^i=\sum_{i\le 0}\langle c_i(\xi),\zeta'\rangle z^i=\langle\xi_{\le 0},\zeta'\rangle$. □

3.2 Macaulay dual

Let $\mathfrak{m}$ be the graded maximal ideal of R. We remark that $R^\vee\simeq\mathrm{Homgr}_K(R,K)\simeq E_R^{\mathrm{gr}}(R/\mathfrak{m})$ the graded injective envelop of the residue field $R/\mathfrak{m}$ of R. $M(d)$ denotes the translation of a graded R-module M by degree $d\in\mathbb{Z}$, i.e., $M(d)_i=M_{d+i}$ for $i\in\mathbb{Z}$ where M_j denotes the degree $j\in\mathbb{Z}$ component of M and by abuse of notation, we denote $M^\vee\simeq\mathrm{Homgr}_K(M,K)$.

Let $I\subseteq R$ be a graded Gorenstein $\mathfrak{m}$-primary ideal. Then the minimal free resolution of $(R/I)^\vee$ has the following form:

$$0\to F_n\to\cdots\to R(d)\to(R/I)^\vee\to 0 \quad \text{(exact)},$$

since $(R/I)^\vee\simeq(R/I)(d)$ for some $d\in\mathbb{Z}$. Taking $(\ )^\vee=\mathrm{Homgr}_K(\_,K)$ on the above exact sequence, we have the minimal injective resolution of R/I:

$$0\to R/I\to R^\vee\to\cdots\to F_{n-1}^\vee\to F_n^\vee\to 0 \quad \text{(exact)}.$$

Hence $R/I\simeq R\cdot F\simeq R/\mathrm{ann}(F)$ for some $F\in R^\vee$, where $\mathrm{ann}(F)=\{g\in R\mid g\cdot F=0\}$. We call $F\in R^\vee$ (or $F^*\in R$) a "Macaulay dual generator" of the $\mathfrak{m}$-primary Gorenstein graded ideal I. Here we remark the following:

$$\mathrm{ann}(f^*)=\{g\in R\mid \mathrm{cont}(g,f)=g\cdot f^*=0\} \quad \text{for} \quad f\in R .$$

Notation 3.2.1. Let $I\subseteq R$ be a $\mathfrak{m}$-primary graded ideal.

(1) $\mathrm{soc}(R/I)=0\underset{R/I}{:}\overline{\mathfrak{m}}=\{\overline{g}\in R/I\mid \overline{g}\overline{\mathfrak{m}}=0\}$ the socle ideal of R/I, where $\overline{\mathfrak{m}}=\mathfrak{m}/I$.

Moreover if I is a $\mathfrak{m}$-primary Gorenstein graded ideal, then $\mathrm{soc}(R/I)\simeq K(-d)$ for some $d\in\mathbb{Z}_{\ge 0}$ and we denote

(2) $\text{soc-deg}(R/I)=d\left(=\max\{i\in\mathbb{Z}\mid (R/I)_i\ne 0\}\right)$.

(3) $F^*(R/I)=F^*$ (up to non-zero constant) if $R/I\simeq R\cdot F$. We say that F^* is a Macaulay dual generator of R/I.

The following lemma asserts that if there exists a surjective morphism between Artinian Gorenstein graded K-algebras of the same socle degree, then the morphism is an isomorphism.

Lemma 3.2.2. *Let* $I\subseteq J\subseteq R$ *be Gorenstein* $\mathfrak{m}$*-primary graded ideals.*

(1) *If* $d=\text{soc-deg}(R/I)=\text{soc-deg}(R/J)$, *then* $I=J$.

(2) $I\subseteq\mathrm{ann}(f^*)$ *and* $\text{soc-deg}(R/I)=\deg f$, *then* $I=\mathrm{ann}(f^*)$.

Proof. (1) By the assumption, we have the surjective morphism $\varphi: R/I \to R/J \to 0$ (exact). Taking out degree d part:

$$0 \to (\ker\varphi)_d \to (R/I)_d \to (R/J)_d \to 0 \ \text{(exact)}.$$

Since $\dim_K (R/I)_d = \dim_K (R/J)_d = 1$, $(\ker\varphi)_d = 0$. If $\ker\varphi \neq 0$, then $\mathrm{soc}(R/I) \subseteq \ker\varphi$, especially $0 \neq \mathrm{soc}(R/I)_d \subseteq (\ker\varphi)_d$.

Hence $\ker\varphi = 0$. This implies $I = J$.

(2) This follows by (1). □

3.3 Macaulay dual generator of $I(a)$

Let us begin with recalling and fixing some notations.

Notation 3.3.1.

(1) $\Psi = \prod_{i=1,\cdots,n-1} (z - x_i) = \sum_{i=0}^{n-1} e'(n-i) z^i$, where we recall that $z = x_n$ and $e'(i) = e_{n-1}(i)$ $(i = 0,\cdots,n-1)$ the singned elementary symmetric plynomial of dgree i in $n-1$ variables.

(2) $\xi_\Psi = \prod_{i=1,\cdots,n-1} (x_i z^{-1} - 1) = \sum_{i=0}^{n-1} e'(i) z^{-i} \in \mathcal{R}$.

(3) $\varDelta = \varDelta_n = \prod_{n \geq i > j \geq 0} (x_i - x_j) \in R$, $\varDelta' = \varDelta_{n-1} = \prod_{n-1 \geq i > j \geq 0} (x_i - x_j) \in R'$ the Vandermonde polynomials.

(4) $W(a) = W_n(a) = e(n)^{a-1} \varDelta \in R$, $W'(a) = W_{n-1}(a) = e'(n-1)^{a-1} \varDelta' \in R'$, where $e(n) = x_1 \cdots x_n$ and $e'(n-1) = x_1 \cdots x_{n-1}$.

Remark 3.3.2. Let a, n be integers with $a \geq 1$ and $n \geq 2$.

(1) $\varDelta = \Psi \varDelta'$.

(2) $W(a) = z^{a-1} e'(n-1)^{a-1} \Psi \varDelta' = z^{a-1} \Psi W'(a)$.

(3) $e'(n-1)\Psi^* = e'(n-1) \prod_{i=1,\cdots,n-1} (z - x_i) = \prod_{i=1,\cdots,n-1} (x_i z^{-1} - 1) = \xi_\Psi$.

(4) Taking dual $(\ )^*$ on the above equation, we have $\Psi = e'(n-1)\xi_\Psi^*$ since $e'(n-1)^* = e'(n-1)^{-1}$.

(5) $W(a) = z^{a-1} e'(n-1)^{a-1} e'(n-1) \xi_\Psi^* \varDelta' = z^{a-1} \xi_\Psi^* e'(n-1)^a \varDelta' = z^{a-1} \xi_\Psi^* W'(a)$.

(6) $J(a) = \sum_{r \geq 0} R\, h(a+r) \subseteq R$, $J'(a) = \sum_{r \geq 0} R\, h(a+r) \subseteq R$.

We recall the notations in section 2 bellow:

$$I(a) = (h(a), h(a+1), \cdots, h(a+n-1)) = (h(a), \cdots, h(a+n-2), z^{a+n-1}) \subseteq R \quad \text{,where } z = x_n.$$

$$I'(a) = (h'(a), \cdots, h'(a+n-2)) = (h'(a), \cdots, h'(a+n-3), x_{n-1}^{a+n-2}) \subseteq R'.$$

Lemma 3.3.3. *Let* $n = 2$ *and* $R = [x_1 = x, x_2 = y]$. *Then* $I(a) \subseteq \mathrm{ann}\left(W(a)^*\right)$.

Proof. Since $I(a) = (h(a), y^{a+1})$, it is enough to show that i) $h(a) \in \mathrm{ann}\left(W(a)^*\right)$ and ii) $y^{a+1} \in \mathrm{ann}\left(W(a)^*\right)$.

i) $\mathrm{cont}\left(h(a),W(a)\right)=\mathrm{cont}\left(\sum_{i=0}^{a}x^{a-i}y^{i},\ x^{a-1}y^{a-1}(y-x)\right)=\left\langle\sum_{i=0}^{a}x^{a-i}y^{i},x^{a-1}y^{a}-x^{a}y^{a-1}\right\rangle$

$$=\left\langle 1,\sum_{i=0}^{a}x^{i-1}y^{a-i}-\sum_{i=0}^{a}x^{i}y^{a-i-1}\right\rangle=\left\langle 1,x^{-1}y^{a}-x^{a}y^{-1}\right\rangle=\pi\left(xy^{-a}-x^{-a}y\right)=0\ .$$

ii) Comparing the degrees with respect to the variable y : $\deg_y y^{a+1}=a+1>\deg_y W(a)=a$, $\mathrm{cont}\left(y^{a+1},W(a)\right)=0$. □

The following is a key lemma for our main result.

Lemma 3.3.4. *We assume that* $n\geq 2$. *Let* $h(a+r)\xi_{\Psi}z^{1-a}=\sum_{p\in\mathbb{Z}}c_p z^{-p}$ *with* $c_p\in R'$ *and* $0\leq r\leq n-1$.

Then $c_p\in I'(a+1)$ *for all* $p\in\mathbb{Z}_{\geq 0}$.

Proof. $h(a+r)\xi_{\Psi}z^{1-a}=\left(\sum_{i\geq 0}h'(a+r-i)z^{i}\right)\left(\sum_{j\in\mathbb{Z}}e'(j)z^{1-a-j}\right)=\sum_{i\geq 0,j\in\mathbb{Z}}h'(a+r-i)e'(j)z^{1-a+i-j}$.

Here we put $1-a+i-j=-p$. Then $i=a+j-p-1\geq 0$ implies $j\geq p+1-a$. Hence we have

$$c_p=\sum_{j\geq p+1-a}h'(r+p+1-j)e'(j)=\begin{cases}\sum_{j\geq 0}h'(r+p+1-j)e'(j)=0 & (p+1-a\leq 0)\\ \sum_{j\geq p+1-a}h'(r+p+1-j)e'(j)=-\sum_{j=0}^{p-a}h'(r+p+1-j)e'(j) & (p+1-a\geq 1)\end{cases}\quad \text{for}\ \ p\geq 0\ ,$$

since $r+p+1\geq 1$.

This implies $c_p\in I'(a+1)$ for all $p\in\mathbb{Z}_{\geq 0}$ since $r+p+1-j\geq a+1$ for $0\leq j\leq p-a$ if $p+1-a\geq 1$. □

Theorem 3.3.5. *Let* $n\geq 2$. *The following hold:*

(1) $I(a)\subseteq\mathrm{ann}\left(W(a)^{*}\right)$ *for any integer* $a\geq 1$.

(2) $I(a)=\left(h(a),h(a+1),\cdots,h(a+n-1)\right)=\mathrm{ann}\left(W(a)^{*}\right)$, i.e., $F^{*}\left(R/I(a)\right)=W(a)$ *the Macaulay dual generator of* $R/I(a)$.

(3) $F^{*}\left(G^{z}\left(R/I(a)\right)\right)=W'(a)z^{a+n-2}$ *the Macaulay dual generator of* $G^{z}\left(R/I(a)\right)$, *where* $z=x_n$.

Proof. (1) It is enough show that $h(a+r)\in\mathrm{ann}\left(W(a)^{*}\right)$, i.e., $\mathrm{cont}\left(h(a+r),W(a)\right)=0$ for all $r\in\mathbb{Z}_{\geq 0}$.

We prove this by induction on number of variables $n\geq 2$. For $n=2$, it is already done by Lemma 3.3.3. Let $n>2$.
Using Remark 3.3.2(5), Remark 3.1.6(1), Lemma 3.1.7, Lemma 3.3.4 and the induction hypothesis, we have

$$\mathrm{cont}\left(h(a+r),W(a)\right)=\left\langle h(a+r),z^{a-1}\xi_{\Psi}^{*}W'(a)\right\rangle=\left\langle h(a+r)\xi_{\Psi}z^{1-a},W'(a)\right\rangle$$

$$=\left\langle\left(h(a+r)\xi_{\Psi}z^{1-a}\right)_{\leq 0},W'(a)\right\rangle=\sum_{p\in\mathbb{Z}_{\geq 0}}\mathrm{cont}\left(c_p,W'(a)\right)z^{-p}=0\ ,$$

where $h(a+r)\xi_{\Psi}z^{1-a}=\sum_{p\in\mathbb{Z}}c_p z^{-p}$ with $c_p\in R'$.

(2) By (1), $I(a)\subseteq\mathrm{ann}\left(W(a)^{*}\right)$. Since $I(a)$ is a $\mathfrak{m}$ -primary complete intersection ideal by Lemma 2.2.1 (3), we have

$$\text{soc-deg}\left(R/I(a)\right)=\sum_{i=0}^{n-1}(a+i-1)=\frac{n(2a+n-3)}{2}=n(a-1)+\frac{n(n-1)}{2}=\deg W(a)\ .$$

Hence $I(a)=\mathrm{ann}\left(W(a)^{*}\right)$ by Lemma 3.2.2 (2). □

(3) From Theorem 2.2.6. and (2), we have

$$F^*\left(G^z\left(R/I(a)\right)\right)=F^*\left(\left(R'/I'\right)\underset{K}{\otimes}\left(K[z]/z^{a+n-1}K[z]\right)\right)=F^*\left(\left(R'/I'\right)\right)\cdot F^*\left(K[z]/z^{a+n-1}K[z]\right)=W'(a)z^{a+n-2}.$$

3.4 Example

We compute some examples in 3 variables case using the Web Interface for Macaulay2 (available at https://www.unimelb-macaulay2.cloud.edu.au/#home). Below, we show the input script for the computation:

```
L1 R=QQ[x,y,z];
L2 b = n-> binomial(n+2,2)-1;h = (n,m) -> sum for i from 0 to m list (monomials(x+y+z)^n)_(0,i);
   h1 = h(1,b(1));h2 = h(2,b(2));h3 = h(3,b(3));h4 = h(4,b(4));h5 = h(5,b(5));h6 = h(6,b(6));h7 = h(7,b(7));h8= h(8,b(8));
L3 f = (z-y)*(z-x);D= f*(y-x);e=x*y*z;g=y-x;W1 = D; W2= D*e; W3= D*e^2; W4= D*e^3; W5= D*e^4; W6= D*e^5;
L4 I1 = ideal(h1,h2,h3); I2 = ideal(h2,h3,h4); I3 = ideal(h3,h4,h5);I4 = ideal(h4,h5,h6); I5 = ideal(h5,h6,h7); I6 = ideal(h6,h7,h8);
```

Explanation:

L1: Declares a polynomial ring in 3 variables.

L2: Generates complete homogeneous symmetric polynomials of degree 1 to 8.

L3: Defines Macaulay dual generators of degree 1 to 6.

L4: Creates ideals $I(a)=\left(h(a),h(a+1),\cdots,h(a+n-1)\right)$ with $1\le a\le 6$.

After inputting the above script, for example:

(1) Input: contract(h3, W3), contract(h4, W3), contract(h5, W3)

Output: (0, 0, 0)

This shows that $\mathrm{cont}\left(h(3),W(3)\right)=\mathrm{cont}\left(h(4),W(3)\right)=\mathrm{cont}\left(h(5),W(3)\right)=0$.

However, we remark that $\mathrm{cont}(f,g)^*=\mathrm{contract}(f,g)$.

(2) Input: ideal(fromDual(W5)) == I5

Output: true

This confirms that $\mathrm{ann}\left(W(5)^*\right)=I(5)$.

Acknowledgement. The author is deeply grateful to Professor Tadahito Harima and Professor Junzo Watanabe for their helpful comments and encouragement. Especially, the proof of Theorem 2.2.6 was provided by J. Watanabe and was introduced by T. Harima in a private discussion. This work is supported by JSPS KAKENHI Grant Number JP20K03508.

(Received: Sep. 30, 2024)
(Accepted: Oct. 28, 2024)

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