**Paper**

# Weak Lefschetz properties of graded modules over a polynomial ring in two variables and the 2-Kronecker quiver

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We give a criterion for an Artinian graded module over a polynomial ring in two variables to have weak Lefschetz property or not by using the classification of indecomposable representations of the 2-Kronecker quiver. As an application, we can recover the well-known result that all principal Artinian graded modules over a polynomial ring in two variables have weak Lefschetz properties.

**Keywords**: Weak Lefschetz property, 2-Kronecker quiver, graded module.

## 1. Introduction

In Ref.(1), Favacchio and Thieu they have studied the weak Lefschetz property of graded modules over a polynomial ring in two variables. In this paper, we also study the weak Lefschetz property of graded modules over a polynomial ring in two variables based on their approach. The new point is that we use the representation of the 2-Kronecker quiver. Since there established the classification of indecomposable representations of the 2-Kronecker quiver, in this point of view, we can describe quite simply whether an Artinian graded module over a polynomial ring in two variables with an algebraically closed field as its coefficient field has weak Lefschetz property or not.

In section 2, we recall the definition of weak Lefschetz property. In section 3, we give a review of representations of 2-Kronecker quiver. After these preparations, we study the weak Lefschetz property of graded modules over a polynomial ring in two variables in section 4.

## 2. Preliminary

We denote the set of integers by $\mathbb{Z}$. Let $R := k[x, y] = \bigoplus_{i\geq 0} R_i$ be the polynomial ring in two variables over an algebraically closed field $k$ with $\mathfrak{m} = \bigoplus_{i\geq 1} R_i$ a graded maximal ideal of $R$, where $R_i \ (i \geq 0)$ and $\mathfrak{m}_i = R_i \ (i \geq 1)$ denote the degree $i$-component of $R$ and $\mathfrak{m}$ respectively. Let denote

$$(i):\mathrm{grmod}R \to \mathrm{grmod}R \ (i \in \mathbb{Z})$$

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the $i$-shift functor defined by $(M(i))_j := M_{i+j}$ for $M = \underset{i\in\mathbb{Z}}{\oplus} M_i \in \mathrm{grmod}R$, where $\mathrm{grmod}R$ the category of finitely generated graded modules over $R$ with standard grading, that is; $x, y$ having degree one.

We mainly work in the subcategory $\mathcal{A} = \text{f.dim-grmod}R$ of $\mathrm{grmod}R$, which is the category of finite dimensional graded modules over $R$. We give the definition of the weak Lefschetz property as follows, we recommend Ref.(2) as a general reference.

**Definition 2.1.** Let $M = \underset{i\in\mathbb{Z}}{\oplus} M_i \in \mathcal{A}$. We say that $M$ has ***weak Lefschetz property*** if the following hold:

There exists $0 \neq l \in R_1$ such that $\times l : M_i \to M_{i+1}$ with $\mathrm{rank}(\times l) = \min\{\dim_k M_i, \dim_k M_{i+1}\}$ for all $i \in \mathbb{Z}$,

where $\times l$ denotes the linear map defined by multiplying by $l$. In this case, we say that $0 \neq l \in R_1$ has ***maximal rank***.

**Notation 2.2.** Let $M = \underset{i\in\mathbb{Z}}{\oplus} M_i \in \mathcal{A}$.

(1) $M_{\geq j} = \underset{i\geq j}{\oplus} M_i \subseteq M$ the graded submodule of $M$ consisting of components of degrees being greater than or equal to $j$.

(2) $\text{deg-Supp}M := \{ i \in \mathbb{Z} \mid M_i \neq 0 \}$.

(3) $w\text{-LF}(M) := \{ [l] \notin \mathbb{P}R_1 \mid l \text{ has maximal rank} \}$ the ***weak Lefschetz locus*** of $M$.

,where we denote the projective space associated with $R_1$ by $\mathbb{P}R_1$ and denote $[l]$ the image of $l$ in $\mathbb{P}R_1$.

**Remark 2.3.** It is well known that $w\text{-LF}(M)$ is Zariski open subset in $\mathbb{P}R_1$, see Ref.(3), of course, it may happen that $w\text{-LF}(M)$ is an empty set.

## 3. Review of representations of 2-Kronecker quiver

In this section we review the classification of finite dimensional indecomposable representations of 2-Kronecker quiver over an algebraically closed field $k$ based on Ref.(4) and Ref.(5). And we also observe the maximal rank locus concerning indecomposable representations.

### 3.1. 2-Kronecker quiver

First, we recall that the 2-Kronecker quiver $\mathbb{K}_2$ is of the following form:

$$\mathbb{K}_2 : \overset{0}{\circ} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \overset{1}{\circ}.$$

Then $V$ a representation of $\mathbb{K}_2$ simply consists of the following data:

$$V = \left( V_0 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} V_1 \right),$$

where $V_i\,(i = 0,1)$ is a finite dimensional vector space over $k$ and $\alpha, \beta$ are linear maps from $V_0$ to $V_1$. This can be seen as a module over the Kronecker algebra $\Lambda = \begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}$, where multiplication is given by the formula

$$\begin{pmatrix} a & 0 \\ v & b \end{pmatrix}\begin{pmatrix} c & 0 \\ w & d \end{pmatrix}=\begin{pmatrix} ac & 0 \\ vc+bw & bd \end{pmatrix}.$$

### 3.2. The regular representations

There are three types of classes of indecomposable finite dimensional $\Lambda$-modules. In this subsection, we will explain about the regular representations.

Let $e \geq 1$ be a positive integer and $\lambda \in k$. The regular representation $\mathcal{R}_\lambda(e)$ is as follows:

$$\mathcal{R}_\lambda(e)=\left( k^e \underset{\beta_{\mathcal{R}_\lambda(e)}=\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}}{\overset{\alpha_{\mathcal{R}_\lambda(e)}=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}}{\rightrightarrows}} k^e \right), \quad \mathcal{R}_\infty(e)=\left( k^e \underset{\beta_{\mathcal{R}_\lambda(e)}=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}}{\overset{\alpha_{\mathcal{R}_\lambda(e)}=\begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}}{\rightrightarrows}} k^e \right).$$

We can see that there are following isomorphisms:

$$\mathcal{R}_\lambda(e) \simeq \left( \mathfrak{m}^{e-1}/\mathfrak{m}^{e} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathfrak{m}^{e}/\left(\mathfrak{m}^{e+1}+(\lambda x-y)^e R\right) \right) (e \geq 1) \text{ with } \alpha=\times x \text{ and } \beta=\times y,$$

$$\mathcal{R}_\infty(e) \simeq \left( \mathfrak{m}^{e-1}/\mathfrak{m}^{e} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathfrak{m}^{e}/\left(\mathfrak{m}^{e+1}+x^e R\right) \right) (e \geq 1) \text{ with } \alpha=\times x \text{ and } \beta=\times y,$$

where we assume that $\mathfrak{m}^0 := R$. Moreover, we remark that there is a natural $k$-algebra homomorphism $\varphi: R \to \Lambda$ as the composition of the following morphisms:

$$R \to R/\mathfrak{m}^2 \simeq \left\{ \begin{pmatrix} c & 0 \\ ax+by & c \end{pmatrix} \middle| a,b,c \in k \right\} \subseteq \left\{ \begin{pmatrix} c & 0 \\ ax+by & c' \end{pmatrix} \middle| a,b,c,c' \in k \right\} \simeq \Lambda = \begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}.$$

This give a restriction functor $\varphi^* = (\ )_R : \operatorname{mod}\Lambda \to \operatorname{mod} R$ associated with $\varphi$ form the category of finite dimensional left $\Lambda$ modules to the category of finitely generated modules over $R$. Also, we can assume that $\varphi: R \to \Lambda$ is a graded $k$-algebra homomorphism. Then we have the functor $\varphi^* = (\ )_R : \operatorname{grmod}\Lambda \to \operatorname{grmod}R$ from form the category of graded finite dimensional right $\Lambda$ modules to the category of graded finitely generated modules over $R$. In this point of view, we have the following lemma:

**Lemma 3.2.1.** *The following holds:*

$$\left(\mathcal{R}_\lambda(e)\right)_R \simeq \left( \mathfrak{m}^{e-1}/\left(\mathfrak{m}^{e+1}+(\lambda x-y)^e R\right) \right)(e-1) \text{ and } \left(\mathcal{R}_\infty(e)\right)_R \simeq \left( \mathfrak{m}^{e-1}/\left(\mathfrak{m}^{e+1}+x^e R\right) \right)(e-1) \quad (e \geq 1), \text{ where } \mathfrak{m}^0 := R.$$

**Lemma 3.2.2.** *The following holds:*

$$w\text{-LF}\left(\left(\mathcal{R}_\lambda(e)\right)_R\right)=\mathbb{P}R_1 \setminus \{[\lambda x-y]\} \quad \textit{and} \quad w\text{-LF}\left(\left(\mathcal{R}_\infty(e)\right)_R\right)=\mathbb{P}R_1 \setminus \{[x]\}.$$

**Proof.** We only prove for $w\text{-LF}\left(\left(\mathcal{R}_\lambda(e)\right)_R\right)$. The other is similar. Let $0 \neq l = ax + by \in R_1$. Thanks to Lemma 3.2.1, $l$ corresponds

to $a\alpha_{\mathcal{R}_\lambda(e)} + b\beta_{\mathcal{R}_\lambda(e)} = a\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} + b\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} = \begin{pmatrix} a+b\lambda & b & 0 & \cdots & 0 \\ 0 & a+b\lambda & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & b \\ 0 & 0 & 0 & \cdots & a+b\lambda \end{pmatrix}$.

If $\lambda = 0$, then $\operatorname{rank}\left(a\alpha_{\mathcal{R}_0(e)} + b\beta_{\mathcal{R}_0(e)}\right) = e$ if and only if $a \neq 0$. If $\lambda \neq 0$, then $\operatorname{rank}\left(a\alpha_{\mathcal{R}_\lambda(e)} + b\beta_{\mathcal{R}_\lambda(e)}\right) = e$ if and only if $a + b\lambda \neq 0$. Hence the assertion follows. □

### 3.3. The preprojectives and preinjectives

The other tow of three types of indecomposable finite dimensional $\Lambda$-modules are preprojective representations and preinjective representations.

Let $d \geq 0$ be a nonnegative integer. The preprojective representation $\mathcal{P}(d)$ and the preinjective representation $\mathcal{I}(d)$ are as follows:

$$\mathcal{P}(d) = \left( k^d \overset{\alpha_{\mathcal{P}(d)}}{\underset{\beta_{\mathcal{P}(d)}}{\rightrightarrows}} k^{d+1} \right), \quad \mathcal{I}(d) = \left( k^{d+1} \overset{\alpha_{\mathcal{I}(d)}}{\underset{\beta_{\mathcal{I}(d)}}{\rightrightarrows}} k^d \right),$$

$$\text{with } \alpha_{\mathcal{P}(d)} = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}}_{d}, \beta_{\mathcal{P}(d)} = \underbrace{\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & \vdots \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}}_{d}, \quad \alpha_{\mathcal{I}(d)} = {}^{\mathrm{tr}}\left(\alpha_{\mathcal{P}(d)}\right) \text{ and } \beta_{\mathcal{I}(d)} = {}^{\mathrm{tr}}\left(\beta_{\mathcal{I}(d)}\right),$$

where ${}^{\mathrm{tr}}A$ is the transpose of matrix $A$.

**Definition 3.3.1.** Let $V = \left( V_0 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} V_1 \right)$ be a $\Lambda$ module. We define $D(V)$ the dual representation of $V$ as follows:

$$D(V) := \operatorname{Hom}_k(V, k) = \left( \operatorname{Hom}_k(V_1, k) \overset{{}^{\mathrm{tr}}\alpha}{\underset{{}^{\mathrm{tr}}\beta}{\rightrightarrows}} \operatorname{Hom}_k(V_0, k) \right).$$

Let $M \in \operatorname{grmod} R$ be a finite dimensional graded module. We denote the dual of $M$ by $M^\vee := \operatorname{Hom}_k(M, k)$.

**Remark 3.3.2.** We can easily see that $D(\mathcal{P}(d)) = \mathcal{I}(d)$ and $D(\mathcal{I}(d)) = \mathcal{P}(d)$, that is; the preprojective representation $\mathcal{P}(d)$ and the preinjective representation $\mathcal{I}(d)$ are dual each other.

With similar observations in subsection 3.2, we have the following lemma:

**Lemma 3.3.3.** *The following holds:*

$$\left(\mathcal{P}(d)\right)_R \simeq \begin{cases} \left(\mathfrak{m}^{d-1}/\mathfrak{m}^{d+1}\right)(d-1) & (\text{if } d \geq 1) \\ k(-1) & (\text{if } d=0) \end{cases} \text{ and } \left(\mathcal{I}(d)\right)_R \simeq \begin{cases} \left(\mathfrak{m}^{d-1}/\mathfrak{m}^{d+1}\right)^{\vee}(-d) & (\text{if } d \geq 1) \\ k & (\text{if } d=0) \end{cases}, \text{ where } \mathfrak{m}^0 := R.$$

**Lemma 3.3.4.** *The following holds:*

$$w\text{-LF}\left(\left(\mathcal{P}(d)\right)_R\right) = w\text{-LF}\left(\left(\mathcal{I}(d)\right)_R\right) = \mathbb{P}R_1.$$

**Proof.** We only prove for $w\text{-LF}\left(\left(\mathcal{P}(d)\right)_R\right)$. The other is similar. Let $0 \neq l = ax+by \in R_1$. Thanks to Lemma 3.3.3, $l$ corresponds

to $$a\alpha_{\mathcal{P}(d)} + b\beta_{\mathcal{P}(d)} = a\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & \vdots \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & \cdots & 0 \\ b & a & \ddots & \vdots \\ 0 & b & \ddots & 0 \\ \vdots & & \ddots & a \\ 0 & 0 & \cdots & b \end{pmatrix}.$$

As far as $ab \neq 0$, we have $\operatorname{rank}\left(a\alpha_{\mathcal{P}(d)} + b\beta_{\mathcal{P}(d)}\right) = d$. So, the assertion follows. □

### 3.4. The classification of finite dimensional indecomposable $\mathbb{K}_2$-representations and their properties

The end of section 3, we will stated well known results concerning the finite dimensional indecomposable $\mathbb{K}_2$-representations and their properties, which are needed later.

**Theorem 3.4.1.** (Ref.(4) Theorem 7.5) *The following hold:*

*(1)* $\Lambda$ *modules* $\mathcal{P}(d), \mathcal{I}(d)\ (d \geq 0)$ *and* $\mathcal{R}_\lambda(e)\ \left(e \geq 1,\ \lambda \in \mathbb{P}_1(k) = k \cup \{\infty\}\right)$ *form a complete set of representatives of the isomorphism classes of indecomposable finite dimensional* $\Lambda$ *modules.*

*(2)* $\dim_k \operatorname{Hom}_\Lambda\left(\mathcal{P}(d), \mathcal{P}(d')\right) = \max\{0, d'-d+1\}$, $\dim_k \operatorname{Hom}_\Lambda\left(\mathcal{I}(d), \mathcal{I}(d')\right) = \max\{0, d-d'+1\}$ $(d, d' \geq 0)$.

*(3)* $\dim_k \operatorname{Hom}_\Lambda\left(\mathcal{P}(d), \mathcal{R}_\lambda(e)\right) = \dim_k \operatorname{Hom}_\Lambda\left(\mathcal{R}_\lambda(e), \mathcal{I}(d)\right) = e$ *and*

*(4)* $\operatorname{Hom}_\Lambda\left(\mathcal{R}_\lambda(e), \mathcal{P}(d)\right) = \operatorname{Hom}_\Lambda\left(\mathcal{I}(d), \mathcal{R}_\lambda(e)\right) = \operatorname{Hom}_\Lambda\left(\mathcal{I}(d), \mathcal{P}(d)\right) = 0$ $\left(d \geq 0,\ e \geq 1,\ \lambda \in \mathbb{P}_1(k)\right)$.

*(5)* $\operatorname{Hom}_\Lambda\left(\mathcal{R}_\lambda(e), \mathcal{R}_{\lambda'}(e')\right) = \delta_{\lambda\lambda'} \min\{e, e'\}$ $\left(e, e' \geq 1,\ \lambda, \lambda' \in \mathbb{P}_1(k)\right)$, *where* $\delta_{\lambda\lambda'}$ *is the Kronecker delta.*

**Remark 3.4.2.** Theorem 3.4.1.(2)-(3) say that nontrivial morphisms go one direction: $\mathcal{P} \to \mathcal{R}_\lambda \to \mathcal{I}$.

$$\mathcal{P}(0) \to \cdots \to \mathcal{P}(d) \to \mathcal{P}(d+1) \to \cdots \underbrace{\boxed{\mathcal{R}_\lambda(e)} \cdots \boxed{\mathcal{R}_{\lambda'}(e')}}_{\lambda,\ \lambda' \in \mathbb{P}_1(k)} \cdots \mathcal{I}(d+1) \to \mathcal{I}(d) \to \cdots \to \mathcal{I}(0).$$

## 4. Application of representations of 2-Kronecker quiver to the weak Lefschetz property

We apply the classification of indecomposable representation of 2-Kronecker quiver to study whether a graded Artinian module has the weak Lefschetz property or not. We denote the category of finite dimensional representations of 2-Kronecker quiver over $k$ by $\mathrm{rep}_k \mathbb{K}_2$ which is equivalent to $\mathrm{mod}\Lambda$ the category of finite dimensional left $\Lambda$ modules. We recall that a morphism from $V=\left(V_0 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} V_1\right)$ to $V'=\left(V'_0 \overset{\alpha'}{\underset{\beta'}{\rightrightarrows}} V'_1\right)$ in $\mathrm{rep}_k \mathbb{K}_2$ is given by a pair of linear maps $f=(f_0, f_1)$ such that the following diagrams commute:

$$\begin{array}{ccc} V_0 & \underset{\alpha}{\to} & V_1 \\ \downarrow^{f_0} & & \downarrow^{f_1} \\ V'_0 & \underset{\alpha'}{\to} & V' \end{array} \quad \text{and} \quad \begin{array}{ccc} V_0 & \underset{\beta}{\to} & V_1 \\ \downarrow^{f_0} & & \downarrow^{f_1} \\ V'_0 & \underset{\beta'}{\to} & V' \end{array}.$$

### 4.1. The frame of a graded module and the weak Lefschetz property

We introduce the frame of a graded module as follows:

**Definition 4.1.1.** Let $M=\underset{i\in\mathbb{Z}}{\oplus} M_i \in \mathrm{grmod}R$. $\mathrm{Fr}(M)=\underset{i\in\mathbb{Z}}{\oplus}\mathrm{Fr}_i(M)\in \mathrm{rep}_k \mathbb{K}_2$ the ***frame*** of $M$ is defined as follows:

$$\mathrm{Fr}_i(M)=\left(M_i \overset{\times x}{\underset{\times y}{\rightrightarrows}} M_{i+1}\right)\in \mathrm{rep}_k \mathbb{K}_2,$$

where we call $\mathrm{Fr}_i(M)$ the ***i-frame*** of $M$.

**Remark 4.1.2.** Let $f: M\to N$ be a morphism in $\mathrm{grmod}R$. We denote $f_i := f|_{M_i}: M_i \to N_i$ the $i$-component of $f$. Then $f$ induces the morphism $\mathrm{Fr}_i(f) := (f_i, f_{i+1}) : \mathrm{Fr}_i(M)\to \mathrm{Fr}_i(N)$ in $\mathrm{rep}_k \mathbb{K}_2$. Hence $\mathrm{Fr}_i\ (i\in\mathbb{Z})$ can be seen an exact functor form $\mathrm{grmod}R$ to $\mathrm{rep}_k \mathbb{K}_2$.

Trough the natural $k$-algebra homomorphism $\varphi: R\to\Lambda$, $\mathrm{Fr}_i(M)$ the $i$-frame of $M$ can be seen a graded $R$ module.

**Lemma 4.1.3.** *Let* $M=\underset{i\in\mathbb{Z}}{\oplus} M_i \in \mathrm{grmod}R$. *We have the following isomorphism:*

$$\left(\mathrm{Fr}_i(M)\right)_R \simeq \left(M_{\geq i}/M_{\geq i+2}\right)(i),$$

*where* $(\ )_R : \mathrm{grmod}\Lambda \to \mathrm{grmod}R$ *denotes the restriction functor.*

**Lemma 4.1.4.** *Let* $M=M_0\oplus M_1 \in \mathcal{A}$ *with* $\deg\text{-}\mathrm{Supp}M \subseteq \{0,1\}$.

*Assume that* $w\text{-LF}\left(\left(\mathrm{Fr}_0(M)\right)_R\right)=w\text{-LF}(M)\neq\varnothing$ *and* $M\simeq M'\oplus N$ *with* $N\neq 0$. *Then the following hold:*

*(1) If* $\dim_k M_0 \leq \dim_k M_1$, *then also we have* $\dim_k N_0 \leq \dim_k N_1$.

*(2) If* $\dim_k M_0 \geq \dim_k M_1$, *then also we have* $\dim_k N_0 \geq \dim_k N_1$.

**Proof.** First we remark that $\left(\mathrm{Fr}_0(M)\right)_R \simeq \left(M_{\geq 0}/M_{\geq 2}\right)(0)\simeq M$ by Lemma 4.1.3.

(1): Let $0\neq l\in R_1$ with $[l]\in w\text{-LF}(M)$. Since $\dim_k M_0\leq \dim_k M_1$, $\times l : M_0\to M_1$ is an injective linear map. Hence

$\times l : N_0 \to N_1$ is also injective. This implies $\dim_k N_0 \le \dim_k N_1$. (2) is similar. □

**Lemma 4.1.5.** *Let* $\mathcal{P}, \mathcal{R}$ *and* $\mathcal{I}$ *denote a finite direct sum of indecomposable preprojective representations, indecomposable regular representations and indecomposable preinjective representations in* $\mathrm{rep}_k \mathbb{K}_2$*, respectively. Then the following hold:*

$$(1)\quad w\text{-LF}\big((\mathcal{P} \oplus \mathcal{R})_R\big) = w\text{-LF}\big((\mathcal{R})_R\big) \neq \varnothing \qquad \text{and} \qquad (2)\quad w\text{-LF}\big((\mathcal{I} \oplus \mathcal{R})_R\big) = w\text{-LF}\big((\mathcal{R})_R\big) \neq \varnothing$$

**Proof.** (1): Let $L := (\mathcal{P})_R$, $N := (\mathcal{R})_R$ and $\mathcal{R} = \bigoplus_{i=1}^{n} \mathcal{R}_{\lambda_i}(e_i)$. Since $\dim_k L_0 \le \dim_k L_1$ and $\dim_k N_0 = \dim_k N_1$, we have $w\text{-LF}\big((\mathcal{P})_R\big) = \mathbb{P}R_1 \neq \varnothing$ and $w\text{-LF}\big((\mathcal{R})_R\big) = \bigcap_{i=1}^{n} w\text{-LF}\big(\big(\mathcal{R}_{\lambda_i}(e_i)\big)_R\big) \neq \varnothing$ by Lemma 3.3.4 and Lemma 3.2.2. Let $M := (\mathcal{P} \oplus \mathcal{R})_R$. Since $\dim_k M_0 \le \dim_k M_1$, $[l] \in w\text{-LF}(M)$ if and only if $\times l : M_0 \to M_1$ is an injective linear map if and only if $[l] \in w\text{-LF}\big((\mathcal{R})_R\big)$. (2) can be shown in the similar way. □

**Lemma 4.1.6.** *Let* $M = M_0 \oplus M_1 \in \mathcal{A}$ *with* $\deg\text{-Supp}M \subseteq \{0, 1\}$*. Then the following are equivalent:*

*(1)* $w\text{-LF}\big((\mathrm{Fr}_0(M))_R\big) = w\text{-LF}(M) \neq \varnothing$ *;*

*(2)* $\mathrm{Fr}_0(M)$ *does not have a nonzero preinjective summand and a nonzero preprojective summand at the same time.*

**Proof.** Since $\mathrm{rep}_k \mathbb{K}_2$ is a Kull-Schmidt category, using classification of indecomposable representations of $\mathbb{K}_2$, we have:

$$\mathrm{Fr}_0(M) \simeq \mathcal{P} \oplus \mathcal{R} \oplus \mathcal{I},$$

where $\mathcal{P}, \mathcal{R}$ and $\mathcal{I}$ denote a finite direct sum of indecomposable preprojective representations, indecomposable regular representations and indecomposable preinjective representations.

(1) $\Rightarrow$ (2): Using Lemma 4.1.4, if $\dim_k M_0 \le \dim_k M_1$, then $\mathrm{Fr}_0(M)$ should be of the following form: $\mathrm{Fr}_0(M) \simeq \mathcal{P} \oplus \mathcal{R}$. On the other hand, if $\dim_k M_0 \ge \dim_k M_1$, then $\mathrm{Fr}_0(M)$ should be of the following form: $\mathrm{Fr}_0(M) \simeq \mathcal{I} \oplus \mathcal{R}$ again by Lemma 4.1.4. Hence (2) follows.

(2) $\Rightarrow$ (1): (2) says that $\mathrm{Fr}_0(M) \simeq \mathcal{P} \oplus \mathcal{R}$ or $\mathrm{Fr}_0(M) \simeq \mathcal{I} \oplus \mathcal{R}$. From Lemma 4.1.5, we have $w\text{-LF}\big((\mathrm{Fr}_0(M))_R\big) \neq \varnothing$. □

**Remark 4.1.7.** First, we remark that $w\text{-LF}(0) = \mathbb{P}R_1$ by definition. Let $M \in \mathcal{A}$. If $w\text{-LF}\big((\mathrm{Fr}_i(M))_R\big) \neq \varnothing$ for all $i \in \mathbb{Z}$, then

$$w\text{-LF}(M) = \bigcap_{i \in \mathbb{Z}} w\text{-LF}\big((\mathrm{Fr}_i(M))_R\big) = \bigcap_{i \in \deg\text{-Supp}M} w\text{-LF}\big((\mathrm{Fr}_i(M))_R\big) \neq \varnothing$$

since $\deg\text{-Supp}M$ is a finite set and each $w\text{-LF}\big((\mathrm{Fr}_i(M))_R\big)$ $(i \in \mathbb{Z})$ is a Zariski open subset of $\mathbb{P}R_1$.

From the above Remark 4.1.7 and Lemma 4.1.6, we get the following theorem:

**Theorem 4.1.8.** *Let* $M \in \mathcal{A}$*. The following are equivalent:*

*(1)* $w$-$\mathrm{LF}(M) \neq \varnothing$

*(2) Each* $\mathrm{Fr}_i(M)\ (i \in \mathbb{Z})$ *does not have a nonzero preinjective summand and a nonzero preprojective summand at the same time.*

**Example 4.1.9.** Let $M := (\mathcal{P}(d))_R \oplus (\mathcal{I}(d'))_R\ (d, d' \geq 0)$. We can see that $w$-$\mathrm{LF}(M) = \varnothing$, that is; $M$ does not have the weak Lefschetz property. In fact, $\dim_k M_0 = \dim_k M_1 = d + d' + 1$. But for any $0 \neq l \in R_1$, the linear map $\times l : M_0 \to M_1$ has nontrivial kernel, since $\times l_{(\mathcal{I}(d'))_R} : k^{d'+1} \to k^{d'}$ the restriction of the linear map $\times l$ is never injective.

### 4.2. In the case of principal graded modules

In this section, we show that every principal Artinian graded module over $R$ has the weak Lefschetz property as an application of Theorem 4.1.8 in the previous section. But in Ref.(6) Proposition 4.4, they have already proved the stronger version of this argument which says that every principal Artinian graded module over $R$ has the strong Lefschetz property.

**Lemma 4.2.1.** *Let* $d \geq 0$ *be a nonnegative integer and let be given the following exact sequence in* $\mathrm{rep}_k \mathbb{K}_2$ *:*

$$0 \to \mathrm{Ker}\,\psi \to \mathcal{P}(d) \to V \to 0 \quad \textit{(exact).}$$

*Then the following are equivalent:*

*(1)* $\mathrm{Ker}\,\psi = 0$*;*

*(2)* $\psi$ *is an isomorphism, that is;* $V \simeq \mathcal{P}(d)$*;*

*(3)* $V$ *has a nonzero preprojective direct summand;*

**Proof.** (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) these implications are trivial, so, we only have to check that (3) $\Rightarrow$ (1).
(3) $\Rightarrow$ (1): If $V \simeq \mathcal{P}(d') \oplus V'$, then $d' \leq d$ by the dimensional reason. Let $\pi : V \to \mathcal{P}(d')$ be the natural projection. Then we have a nonzero homomorphism $0 \neq \pi \circ \psi : \mathcal{P}(d) \to \mathcal{P}(d') \in \mathrm{Hom}_\Lambda(\mathcal{P}(d), \mathcal{P}(d'))$. By Theorem 3.4.1 (2), $d \leq d'$ since $\mathrm{Hom}_\Lambda(\mathcal{P}(d), \mathcal{P}(d')) \neq 0$. After all, we have $d' = d$, so $V \simeq \mathcal{P}(d)$. □

Let $M \in \mathcal{A}$. The Hilbert function $\mathrm{HF}_M$ is defined by $\mathrm{HF}_M(i) := \dim_k M_i\ (i \in \mathbb{Z})$. We say that $M$ has ***unimodal*** Hilbert function if there exists an integer $i_0 \in \mathbb{Z}$ such that $\mathrm{HF}_M(i_0) > \mathrm{HF}_M(i_0 + 1)$, then $\mathrm{HF}_M(i) \geq \mathrm{HF}_M(i+1)$ for all $i \geq i_0$. It is well know that if $M \in \mathcal{A}$ has the strong Lefschetz property, then $M$ has unimodal Hilbert function.

**Theorem 4.2.2.** *Every principal Artinian graded module* $M$ *over* $R$ *has the weak Lefschetz property, moreover* $M$ *has unimodal Hilbert function.*

**Proof.** If $M = 0$, there is nothing to prove, so we can assume that $M \neq 0$. Furthermore, since $M$ is a principal Artinian grade module, we can assume that $M$ is generated by a degree 0 element after an appropriate degree shifting if it is needed. Then we have the following exact sequence in $\mathrm{grmod} R$ :

$$0 \to I \to R \to M \to 0 \quad \text{(exact),}$$

where $I$ is a proper homogeneous ideal of $R$. Applying the exact functor $\mathrm{Fr}_i$ to the above exact sequence, we have the following exact sequence in $\mathrm{rep}_k \mathbb{K}_2$:

$$0 \to \mathrm{Fr}_i(I) \to \mathrm{Fr}_i(R) \to \mathrm{Fr}_i(M) \to 0 \ (i \in \mathbb{Z}) \ \text{(exact)}.$$

Here we remark that $\mathrm{Fr}_i(R)=0$ if $i<0$ and $\mathrm{Fr}_i(R) \simeq \mathcal{P}(i)$ for $i \geq 0$. Moreover, if $\mathrm{Fr}_{i_0}(I) \neq 0$ for some integer $i_0 > 0$, then $\mathrm{Fr}_i(I) \neq 0$ for any $i \geq i_0$. Using Lemma 4.2.1, we can see that there exists an integer $i_0 \geq 0$ such that

$$\mathrm{Fr}_i(M) = \begin{cases} \mathrm{Fr}_i(M) \simeq \mathcal{P}(i) & (0 \leq i < i_0) \\ \mathrm{Fr}_i(M) \text{ has no nonzero preprojective summand} & (i \geq i_0) \end{cases}.$$

Applying Theorem 4.1.8, we can see that $w\text{-LF}(M) \neq \varnothing$, that is; $M$ has the weak Lefschetz property.

If $i \geq i_0$ then $\dim_k \mathrm{Fr}_i(M)_0 = \dim_k M_i \geq \dim_k \mathrm{Fr}_i(M)_1 = \dim_k M_{i+1}$, since $\mathrm{Fr}_i(M)$ has no nonzero preprojective summand, that is; $\mathrm{Fr}_i(M)$ has the following form: $\mathrm{Fr}_i(M) \simeq \mathcal{I} \oplus \mathcal{R}$, where $\mathcal{I}$ and $\mathcal{R}$ denote a finite direct sum of indecomposable preinjective representations and indecomposable regular representations in $\mathrm{rep}_k \mathbb{K}_2$, respectively. On the other hand, if $i < i_0$, $\dim_k \mathrm{Fr}_i(M)_0 = \dim_k \mathcal{P}(i)_0 = \dim_k M_i \leq \dim_k \mathrm{Fr}_i(M)_1 = \dim_k \mathcal{P}(i)_1 = \dim_k M_{i+1}$. Hence $M$ has unimodal Hilbert function. □

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