Paper

Almost regular sequences on graded modules associated with a subfunctor of the 0-th local cohomology functor II

Satoru Isogawa[1*]

This is the last part of this two-part series of papers, here, we introduce the γ- regular sequence in the category of finitely generated graded modules over a polynomial ring whose coefficient field is an infinite field. For a nonzero principal graded module, its first syzygy ideal being a completely $\mathfrak{m}$- full, is equivalent to that the principal module has a γ- regular sequence whose length is equal to the Kull dimension of the base polynomial ring. This is the result of Watanabe and Harima, see Ref(1). We extend the result as follows. For graded modules whose first syzygies are generated by elements of the same degree, to have a component-wise linear syzygy so actually to have linear syzygy is equivalent to that there exists a γ- regular sequence whose length is equal to the Kull dimension of the base ring.

Keywords: Almost regular sequences, γ- regular sequences, Castelnuovo-Mumford regularity, linear resolutions, $\mathfrak{m}$- full ideals.

1. Introduction

This is the last part of this two-part series of papers. Here, we introduce '*γ -regular elements, γ - regular sequences and γ - depth's*' on graded modules given by strengthening the notion weak γ -regular, which we studied in the first part of this series of papers, to γ -regular.

For a nonzero principal graded module, its first syzygy ideal being a $\mathfrak{m}$ -full ideal, see Ref.(1), is equivalent to that the principal module has a γ -regular element in our terminology. In Ref.(2), they proved surprising result that completely $\mathfrak{m}$ -full ideals are just the ideals having component-wise linear resolutions, also see Ref.(3) or Ref.(4). The first syzygy ideal of a nonzero principal graded module is a completely $\mathfrak{m}$ -full ideal if and only if the γ -depth of the principal graded module is equal to the Kull dimension of base polynomial ring.

To extend this result to arbitrary modules is our main motivation through this series of papers. Toward solving this problem, first, we give several fundamental results on γ -regular elements, γ -sequences and γ -depth base on the study of weak γ -regular elements, weak γ - regular sequences and weak γ - depth's on graded modules in the first part of this series of papers. As a consequence, we give a partial answer to the problem only in the case of modules whose first syzygies are generated by elements of the same degree (see Theorem 6.4).

In this paper, we follow the notations and definitions sated in the first part of this series of papers, see Ref(5).

* Faculty of Liberal Arts
2627 Hirayama-Shinmachi Yatsushiro-shi Kumamoto, Japan 866-8501
E-mail address: isogawa@kumamoto-nct.ac.jp (S. Isogawa)

2. Preliminary

In this section, we add more notations and definitions needed later.

Notation 2.1. Let $M=\bigoplus_{i\in\mathbb{Z}} M_i \in \mathcal{A}$ and $0\neq z\in R_1$.

(1) $M_{\geq j}=\bigoplus_{i\geq j} M_i \subseteq M$ the graded submodule of M consisting of components of degrees being greater than or equal to *j*.

(2) $M_{\langle i\rangle}=RM_i\subseteq M$ the graded submodule of M generated by the degree *i* component M_i.

(3) $\deg\xi := i$ the degree of ξ for a nonzero homogeneous element $0\neq\xi\in M_i$ $(i\in\mathbb{Z})$ if it exists.

(4) $\text{deg-Supp}M := \{i\in\mathbb{Z} \mid M_i\neq 0\}$.

(5) $\sup M := \max\{i\in\mathbb{Z} \mid M_i\neq 0\}$ if $M\neq 0$, otherwise, $\sup 0 := -\infty$.

(6) $\inf M := \inf\{i\in\mathbb{Z} \mid M_i\neq 0\}$, if $M\neq 0$, otherwise, $\inf 0 := \infty$.

(7) $\overline{R}=\overline{R}^{(z)} := R/zR$ and $\overline{M}=\overline{M}^{(z)} := \overline{R}\otimes_R M$, unless otherwise mentioned.

(8) $\operatorname{length}M$ denote the length of M, in our setting, $\operatorname{length}M=\dim_k M$ the dimension of k-vector space of M through the natural inclusion of algebras: $k\to R$.

(9) Let $N\subseteq M$ be a graded submodule. $\operatorname{Sat}_M N := \{\xi\in M \mid \mathfrak{m}^i\xi\subseteq N \text{ for some } i\geq 0\}$, which is called the saturation of N in M. By the definition, we remark that $\operatorname{Sat}_M N=\operatorname{Sat}_M \mathfrak{m}^i N$ for any integer $i\geq 0$, where $\mathfrak{m}^0 := R$.

(10) $\pi_M=\pi_M^R : F_M=F_M^R\to M$ denote a minimal graded free cover of M over R.
Of course, there are many choices of a minimal graded free cover of M, but π_M^R is determined unique up to isomorphism as explained bellow.

Let $\pi_M : F_M\to M$ be a minimal grade free cover of $M\in\mathcal{A}$, then we have the following exact sequence:

$$0\to\Omega_R^1 M\to F_M\xrightarrow[\pi_M]{} M\to 0 \quad \text{(exact)},$$

where $\Omega_R^1 M$ denote the first syzygy module of M over R. We remark that the above short exact sequence is unique up to isomorphism in the following sense, especially $\Omega_R^1 M$ is unique up to isomorphism. In fact, if $\pi' : F'\to M$ another minimal grade free cover of M, then using the grade Nakayama's Lemma, we have the following commutative diagram with exact rows and all vertical morphisms being isomorphisms:

$$\begin{array}{ccccccccc} 0 & \to & \Omega_R^1 M & \to & F_M & \xrightarrow[\pi_M]{} & M & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \operatorname{Ker}\pi' & \to & F' & \xrightarrow[\pi']{} & M & \to & 0 \end{array}.$$

So, the following notations make sense.

Notation 2.2. Let $0\to\Omega_R^1 M\to F_M\xrightarrow[\pi_M]{} M\to 0$ be an exact sequence with $\pi_M : F_M\to M$ a minimal grade free cover of $M\in\mathcal{A}$.

(1) $M^{\#}=M_R^{\#} := F_M/\mathfrak{m}\Omega_R^1 M$ and $M^{\#*i}=M_R^{\#*i} := F_M/\mathfrak{m}^i\Omega_R^1 M$ for integers $i\geq 0$, where we assume that $\mathfrak{m}^0 := R$.

(2) $M^{[j]}=M_R^{[j]} := F_M/\left(\Omega_R^1 M\right)_{\langle j\rangle}$ and $M^{[\geq j]}=M_R^{[\geq j]} := F_M/\left(\Omega_R^1 M\right)_{\langle\geq j\rangle}$ for $j\in\mathbb{Z}$.

These notations depend on a choice of minimal grade free cover, but they are unique up to isomorphism.

3. γ-regular elements and sequences

First let $0 \to \Omega_R^1 M \to F_M \underset{\pi_M}{\to} M \to 0$ be an exact sequence with $\pi_M^R = \pi_M : F_M \to M$ a minimal grade free cover over R and let $\overline{R} := R/zR$ with $0 \neq z \in R_1$. Then we have an exact sequence $\overline{R} \underset{R}{\otimes} \Omega_R^1 M \to \overline{F_M} \underset{\overline{R} \otimes_R \pi_M}{\to} \overline{M} \to 0$. Since $\overline{\pi_M^R} := \overline{R} \underset{R}{\otimes} \pi_M^R$ is a minimal graded free cover of $\overline{M}$ over $\overline{R}$, we can see that $\pi_{\overline{M}}^{\overline{R}} = \overline{\pi_M^R}$. Moreover, the first syzygy module of $\overline{M}$ over $\overline{R}$ has the following form:

$$\Omega_{\overline{R}}^1 \overline{M} = \ker\left(\overline{R} \underset{R}{\otimes} \pi_M\right) \simeq \left(\Omega_R^1 M + zF_M\right) / zF_M .$$

So, we have

$$\Omega_{\overline{R}}^1 \overline{M} \Big/ \mathfrak{m}\Omega_{\overline{R}}^1 \overline{M} \simeq \left(\Omega_R^1 M + zF_M\right)/zF_M \Big/ \left(\mathfrak{m}\Omega_R^1 M + zF_M\right)/zF_M \simeq \left(\Omega_R^1 M + zF_M\right) \Big/ \left(\mathfrak{m}\Omega_R^1 M + zF_M\right).$$

Secondly, there is a natural short exact sequence $0 \to \mathfrak{m}\Omega_R^1 M \to F_M \underset{\pi_{M_R^\#}}{\to} M_R^\# \to 0$, $\pi_{M_R^\#} : F_M \to M_R^\# \to 0$ being a graded minimal free cover of $M_R^\#$ over R. We get the exact sequence $\overline{R} \underset{R}{\otimes} \left(\mathfrak{m}\Omega_R^1 M\right) \to \overline{F_M} \underset{\overline{R} \otimes_R \pi_{M_R^\#}}{\to} \overline{M_R^\#} \to 0$, by applying $\overline{R} \underset{R}{\otimes} \_$ to the above short exact sequence. Since $\overline{\pi_{M_R^\#}} := \overline{R} \underset{R}{\otimes} \pi_{M_R^\#}$ is a minimal graded free cover of $\overline{M_R^\#} \simeq F_M \Big/ \left(\mathfrak{m}\Omega_R^1 M + zF_M\right) \simeq \overline{M}_{\overline{R}}^\#$ over $\overline{R}$, we can see that $\overline{\pi_{M_R^\#}} = \overline{\pi_{M_R^\#}^R} = \pi_{\overline{M}_{\overline{R}}^\#}^{\overline{R}} = \pi_{\overline{M}_{\overline{R}}^\#}$.

Finally, there is a natural short exact sequence $0 \to \Omega_R^1 M \big/ \mathfrak{m}\Omega_R^1 M \to M^\# \underset{\rho_{M^\#}}{\to} M \to 0$. Applying $\overline{R} \underset{R}{\otimes} \_$, we have the following exact sequence:

$$\overline{R} \underset{R}{\otimes} \left(\Omega_R^1 M \big/ \mathfrak{m}\Omega_R^1 M\right) \to \overline{M^\#} \underset{\overline{R} \otimes \rho_{M^\#}}{\to} \overline{M} \to 0 .$$

Moreover, we can see that

$$\operatorname{Ker}\left(\overline{R} \otimes \rho_{M^\#} : \overline{M}^\# = \overline{M^\#} \to \overline{M}\right) \simeq \left(\Omega_R^1 M + zF_M\right) \Big/ \left(\mathfrak{m}\Omega_R^1 M + zF_M\right) \simeq \Omega_{\overline{R}}^1 \overline{M} \Big/ \mathfrak{m}\Omega_{\overline{R}}^1 \overline{M} \simeq {}^*\mathrm{Tor}_1^{\overline{R}}\left(k, \overline{M}\right).$$

Remark 3.1. From the above observation, we have the following short exact sequence:

$$0 \to {}^*\mathrm{Tor}_1^{\overline{R}}\left(k, \overline{M}\right) \to \overline{M_R^\#} \underset{\overline{R} \otimes_R \rho_{M^\#}}{\to} \overline{M} \to 0 \quad \text{(exact).}$$

Definition 3.2. Let $0 \neq z \in R_1, M \in \mathcal{A}$. We call that z is a ***γ-regular element*** on M, a ***M-γ-regular element*** or a ***γ-'non zero-divisor'*** on M if the following condition holds:

$$0 \to {}^*\mathrm{Tor}_1^R\left(R/zR, M\right) \xrightarrow{{}^*\mathrm{Tor}_1^R\left(\varphi_{R/zR}, M\right)} {}^*\mathrm{Tor}_1^R\left(k, M\right) \quad \text{(exact),}$$

where $\varphi_{R/zR} : R/zR \to k$ is a natural projection and two functors ${}^*\mathrm{Tor}_1^R(R/zR, \_)$, ${}^*\mathrm{Tor}_1^R(k, \_)$ are graded torsion functors.

From the definition of γ -regular element, we have the following lemma.

Lemma 3.3. *Let $M \in \mathcal{A}$ and $0 \neq z \in R_1$ is a γ -regular element on M . Then the following holds:*

$$\operatorname{deg-Supp}(\gamma(M)(-1)) \subseteq \operatorname{deg-Supp}\left(k \otimes_R \Omega_R^1 M\right).$$

Proof. By the assumption we have $0 \to {}^*\mathrm{Tor}_1^R\left(R/zR, M\right) \to {}^*\mathrm{Tor}_1^R(k, M)$ (exact) where ${}^*\mathrm{Tor}_1^R(k, M) \simeq k \otimes_R \Omega_R^1 M$ and ${}^*\mathrm{Tor}_1^R\left(R/zR, M\right) \simeq \gamma_z(M)(-1) \simeq \gamma(M)(-1)$ since $\mathfrak{m}\,{}^*\mathrm{Tor}_1^R\left(R/zR, M\right) = 0$. Hence the assertion follows. □

Definition 3.4. Let $M \in \mathcal{A}$. We define $\gamma\text{-nzd}(M)$ the γ *-'non zero-divisor' locus* of M in $\mathbb{P}R_1$ as follows:

$$\gamma\text{-nzd}(M) := \left\{[z] \in \mathbb{P}R_1 \,\middle|\, z \text{ is a } \gamma\text{-regular element on } M\right\}.$$

Definition 3.5. A sequence $\underline{z} = z_1, \cdots, z_r$ of elements in R_1 is called a γ ***-regular sequence*** on M , M-γ- ***regular sequence*** or, more simply, M-γ ***-sequence*** of length r if the following conditions are satisfied:

(1) The set $\{z_1, \cdots, z_r\}$ is linearly independent over the field k .

(2) First, z_1 is a γ -regular elements on M and if $r \geq 2$,

then z_i is a γ -regular elements on $M/(z_1, \cdots, z_{i-1})M$ for $i = 2, \cdots, r$.

Remark 3.6. Two conditions (1) and (2) in the above definition are equivalent to the following condition (3):

(3) $\overline{z_i}^{(i-1)} \in \left(\overline{R}^{(i-1)}\right)_1$ is a γ -regular elements on $\overline{M}^{(i-1)}$ for $i = 1, \cdots, r$,

where $\overline{R}^{(0)} := R$, $\overline{R}^{(i)} := R/(z_1, \cdots, z_i)R$, $\overline{M}^{(i)} := M \otimes_R \overline{R}^{(i)}$ and $\overline{z_i}^{(i-1)}$ denoting the image of z_i in $\overline{R}^{(i-1)}$ for $i = 1, \cdots, r$.

4. Properties of γ -regular elements

For $M \in \mathcal{A}$, we have the following commutative diagrams with exact rows and all vertical morphisms being isomorphisms:

$$\begin{array}{ccccccccc} 0 & \to & {}^*\mathrm{Tor}_1^R(k, M) & \to & M^{\#} & \underset{\rho_{M^{\#}}}{\to} & M & \to & 0 \\ & & \downarrow & & \| & & \| & & \\ 0 & \to & \Omega_R^1 M / \mathfrak{m}\Omega_R^1 M & \to & F_M / \mathfrak{m}\Omega_R^1 M & \underset{\rho}{\to} & F_M / \Omega_R^1 M & \to & 0 \end{array},$$

where ρ is the natural projection. Applying two functors $k \otimes_R \_$ and $\overline{R} \otimes_R \_$ where $\overline{R} := R/zR$ with $0 \neq z \in R_1$ to the top row of the above commutative diagram, we have the following commutative diagrams with exact rows:

$$\begin{array}{ccccccccccccc}
0 \to {}^*\mathrm{Tor}_1^R\left(\overline{R}, \mathrm{Tor}_1^R(k, M)\right) \to {}^*\mathrm{Tor}_1^R\left(\overline{R}, M^\#\right) \to & {}^*\mathrm{Tor}_1^R\left(\overline{R}, M\right) & \underset{\delta}{\to} & {}^*\mathrm{Tor}_1^R(k, M) & \to & \overline{M}^\# & \underset{\overline{R}\otimes\rho_{M^\#}}{\to} & \overline{M} & \to & 0 \\
& \downarrow\tau & & \| & & \downarrow & & \downarrow & & \\
& {}^*\mathrm{Tor}_1^R(k, M) & \underset{\delta'}{\to} & {}^*\mathrm{Tor}_1^R(k, M) & \to & k\underset{R}{\otimes}M^\# & \underset{k\otimes\rho_{M^\#}}{\to} & k\underset{R}{\otimes}M & \to & 0
\end{array},$$

where $\tau := {}^*\mathrm{Tor}_1^R\left(\varphi_{R/zR}, M\right)$. We can see that $k\otimes\rho_{M^\#}$ is an isomorphism, so δ' is an epimorphism, especially an isomorphism since δ' is an endomorphism of the finite dimensional vector space ${}^*\mathrm{Tor}_1^R(k, M)$. Since $\delta = \delta'\circ\tau$, where δ' is an isomorphism, we have $\mathrm{Coker}\,\tau \simeq \mathrm{Coker}\,\delta$. On the other hand, by Remark 6.3, $\mathrm{Coker}\,\delta \simeq \mathrm{Ker}\left(\overline{R}\otimes\rho_{M^\#}\right) \simeq \mathrm{Tor}_1^{\overline{R}}\left(k, \overline{M}\right)$ we have the following exact sequence:

$$\boxed{\begin{array}{ccccccc}
0 \to & {}^*\mathrm{Tor}_1^R\left(\overline{R}, \mathrm{Tor}_1^R(k, M)\right) & \to & {}^*\mathrm{Tor}_1^R\left(\overline{R}, M^\#\right) & \to & {}^*\mathrm{Tor}_1^R\left(\overline{R}, M\right) & \underset{\tau}{\to} {}^*\mathrm{Tor}_1^R(k, M) \to \mathrm{Coker}\,\tau \simeq {}^*\mathrm{Tor}_1^{\overline{R}}\left(k, \overline{M}\right) \to 0 \\
& \| & & \| & & \| & \\
& {}^*\mathrm{Tor}_1^R(k, M)(-1) & & \gamma_z\left(M^\#\right)(-1) & & \gamma_z(M)(-1) &
\end{array}}$$

For $M \in \mathcal{A}$, let denote $\beta_i^R(M) := \dim_k {}^*\mathrm{Tor}_i^R(k, M)$ the i-th total Betti number of M over R. As an immediate consequence of the above exact sequence, we have the following proportion:

Proposition 4.1. *Let* $0 \neq z \in R_1, M \in \mathcal{A}$. *The following conditions are equivalent:*

(1) z *is a* M*-*γ*-regular element;*

(2) $\beta_1^R(M) = \mathrm{length}\,\gamma_z\left(M^\#\right)$*;*

(3) z *is a weak* $M^\#$*-*γ*- regular element and* $\beta_1^R(M) = \mathrm{length}\,\gamma\left(M^\#\right)$*;*

(4) $\beta_1^R(M) - \beta_1^{\overline{R}}\left(\overline{M}\right) = \mathrm{length}\,\gamma_z(M)$.

Proof. The equivalences $(1) \Leftrightarrow (2) \Leftrightarrow (4)$ are clear by the exact sequence stated above. We only prove $(2) \Leftrightarrow (3)$.

$(2) \Rightarrow (3)$: $\beta_1^R(M) = \mathrm{length}\,\gamma_z\left(M^\#\right)$ implies $\gamma_z\left(M^\#\right)(-1) \simeq {}^*\mathrm{Tor}_1^R(k, M)(-1)$ so $\gamma_z\left(M^\#\right) = \gamma\left(M^\#\right)$ since $\mathfrak{m}\gamma_z\left(M^\#\right) = 0$.

Therefore, z is a weak $M^\#$-γ- regular element and $\beta_1^R(M) = \mathrm{length}\,\gamma_z\left(M^\#\right) = \mathrm{length}\,\gamma\left(M^\#\right)$.

$(3) \Rightarrow (2)$: Since z is a weak $M^\#$-γ- regular element, we have $\gamma_z\left(M^\#\right) = \gamma\left(M^\#\right)$.

Hence, $\beta_1^R(M) = \mathrm{length}\,\gamma\left(M^\#\right) = \mathrm{length}\,\gamma_z\left(M^\#\right)$. □

Corollary 4.2. *Let* $M \in \mathcal{A}$. *The following holds:*

$$\gamma\text{-nzd}(M)=\begin{cases} w\text{-}\gamma\text{-nzd}(M^{\#}) & \left(\text{if } \beta_1^R(M)=\text{length}\,\gamma(M^{\#})\right) \\ \varnothing & (\text{otherwise}) \end{cases}.$$

Especially $\gamma\text{-nzd}(M)$ *is a Zariski open set in* $\mathbb{P}R_1$.

Proposition 4.3. *Let* $M\in\mathcal{A}$. *If* $0\neq z\in R_1$ *is a* γ- *regular element on* M, *then the following hold:*

$$\overline{R}\underset{R}{\otimes}\Omega_R^1 M\simeq\gamma(M)(-1)\oplus\Omega_{\overline{R}}^1\left(\overline{M}\right),$$

where $\overline{R}:={R}/{zR}$ *and* $\overline{M}:={M}/{zM}$ *,especially,* $\deg\text{-Supp}\left(k\underset{R}{\otimes}\Omega_R^1 M\right)=\deg\text{-Supp}\left(\gamma(M)(-1)\right)\cup\deg\text{-Supp}\left(k\underset{R}{\otimes}\Omega_{\overline{R}}^1\overline{M}\right)$.

Proof. Let $\pi_M:F_M\to M$ be a graded minimal free cover, then we have the following exact sequence:

$$0\to\Omega_R^1 M\to F_M\underset{\pi_M}{\to}M\to 0 \quad (\text{exact}).$$

Taking tensor functor $\overline{R}\underset{R}{\otimes}\_\_$ on the above exact sequence yields the following exact sequence:

$$0\to{}^*\text{Tor}_1^R\left(\overline{R},M\right)\underset{\delta}{\to}\overline{R}\underset{R}{\otimes}\Omega_R^1 M\to\overline{R}\underset{R}{\otimes}F_M\underset{\overline{R}\underset{R}{\otimes}\pi_M}{\to}\overline{R}\underset{R}{\otimes}M\to 0 \quad (\text{exact}).$$

Since $\text{Ker}\left(\overline{R}\underset{R}{\otimes}\pi_M\right)\simeq\Omega_{\overline{R}}^1\overline{M}$, from the above exact sequence, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & {}^*\text{Tor}_1^R\left(\overline{R},M\right) & \underset{\delta}{\to} & \overline{R}\underset{R}{\otimes}\Omega_R^1 M & \to & \Omega_{\overline{R}}^1\overline{M} & \to & 0 \\ & & \| & & \downarrow^{\rho} & & & & \\ 0 & \to & {}^*\text{Tor}_1^R\left(\overline{R},M\right) & \underset{{}^*\text{Tor}_1^R\left(\pi_{\overline{R}},M\right)}{\to} & {}^*\text{Tor}_1^R(k,M)\simeq k\underset{R}{\otimes}\Omega_R^1 M & & & & \end{array},$$

where the right vertical morphism ρ is a natural projection. The morphism δ is split since ${}^*\text{Tor}_1^R\left(\pi_{R/zR},M\right)$ is a split injective morphism between finite dimensional vector spaces.

Hence we have

$$\overline{R}\underset{R}{\otimes}\Omega_R^1 M\simeq{}^*\text{Tor}_1^R\left(\overline{R},M\right)\oplus\Omega_{\overline{R}}^1\overline{M}\simeq\gamma_z(M)(-1)\oplus\Omega_{\overline{R}}^1\overline{M}\simeq\gamma(M)(-1)\oplus\Omega_{\overline{R}}^1\overline{M}. \quad \square$$

5. Regularity and γ-regular sequence

First, we recall the definition of the Castelnuovo-Mumford regularity.

Definition 5.1. (See Ref.(4)) Let $M\in\mathcal{A}$. The Castelnuovo-Mumford regularity or simply regularity $\text{reg}_R M$ is define as follows:

$$\text{reg}_R M:=\sup\left\{j\in\mathbb{Z}\,\middle|\,{}^*\text{Tor}_i^R(k,M)_{i+j}\neq 0\right\}.$$

Especially, $\text{reg}_R 0:=-\infty$.

Proposition 5.2. *Let* $M \in \mathcal{A}$ *. If* $0 \neq \underline{z} = z_1, \cdots, z_r \in R_1 \ (r \geq 1)$ *is a* M*-*γ*- sequence, then the following hold:*

$$\operatorname{reg}_R \Omega_R^1 M = \max\left\{ \sup\left(k \underset{R}{\otimes} \Omega_R^1 M \right),\ \operatorname{reg}_{\overline{R}} \Omega_{\overline{R}}^1 \left(\overline{M} \right) \right\},$$

where $\overline{R} := R \big/ (z_1, \cdots, z_r) R$ *and* $\overline{M} := \overline{R} \underset{R}{\otimes} M$ *, especially the inequality* $\operatorname{reg}_{\overline{R}} \Omega_{\overline{R}}^1 \left(\overline{M} \right) \leq \operatorname{reg}_R \Omega_R^1 M$ *holds.*

Proof. We prove this induction on r the length of M-γ- sequence.

If $r = 1$, then we remark that the following hold:

$$\operatorname{reg}_R \Omega_R^1 M = \operatorname{reg}_{\overline{R}} \left(\overline{R} \underset{R}{\otimes} \Omega_R^1 M \right).$$

Since z_1 is $\Omega_R^1 M$ -regular element, so $\beta_{ij}^R \left(\Omega_R^1 M \right) = \beta_{ij}^{\overline{R}} \left(\overline{R} \underset{R}{\otimes} \Omega_R^1 M \right)$ for any nonnegative integer i and any integer j, where $\beta_{ij}^R \left(\Omega_R^1 M \right)$ and $\beta_{ij}^{\overline{R}} \left(\overline{R} \underset{R}{\otimes} \Omega_R^1 M \right)$ denote the graded Betti numbers of $\Omega_R^1 M$ over R and $\overline{R} \underset{R}{\otimes} \Omega_R^1 M$ over $\overline{R}$ respectively.

By Proposition 6.11, $\operatorname{reg}_R \Omega_R^1 M = \operatorname{reg}_{\overline{R}} \left(\overline{R} \underset{R}{\otimes} \Omega_R^1 M \right) = \operatorname{reg}_{\overline{R}} \left(\gamma(M)(-1) \oplus \Omega_{\overline{R}}^1 \left(\overline{M} \right) \right) = \max\left\{ \operatorname{reg}_{\overline{R}} \gamma(M)(-1),\ \operatorname{reg}_{\overline{R}} \Omega_{\overline{R}}^1 \left(\overline{M} \right) \right\}$, and

$$\operatorname{reg}_{\overline{R}} \gamma(M)(-1) = \sup \gamma(M)(-1) = \sup \gamma(M) + 1 \leq \sup\left(k \underset{R}{\otimes} \Omega_R^1 M \right),$$

since $\gamma(M)(-1)$ is a direct sum of copies of residue field k with $\operatorname{reg}_{\overline{R}} k = 0$ up to grading.

As a consequence, we have the following inequality:

$$\sup\left(k \underset{R}{\otimes} \Omega_R^1 M \right) \leq \operatorname{reg}_R \Omega_R^1 M = \max\left\{ \sup \gamma(M) + 1,\ \operatorname{reg}_{\overline{R}} \Omega_{\overline{R}}^1 \left(\overline{M} \right) \right\} \leq \max\left\{ \sup\left(k \underset{R}{\otimes} \Omega_R^1 M \right),\ \operatorname{reg}_{\overline{R}} \Omega_{\overline{R}}^1 \left(\overline{M} \right) \right\}.$$

This is equivalently to say that

$$\operatorname{reg}_R \Omega_R^1 M = \max\left\{ \sup\left(k \underset{R}{\otimes} \Omega_R^1 M \right),\ \operatorname{reg}_{\overline{R}} \Omega_{\overline{R}}^1 \left(\overline{M} \right) \right\}.$$

If $r \geq 2$, then from the above argument, we can see that

$$\operatorname{reg}_R \Omega_R^1 M = \max\left\{ \sup\left(k \underset{R}{\otimes} \Omega_R^1 M \right),\ \operatorname{reg}_{R'} \Omega_{R'}^1 (M') \right\}$$

where $R' = R \big/ z_1 R$, $M' = M \big/ z_1 M$.

On the other hand, let $\overline{z_2}, \cdots, \overline{z_r} \in R_1'$ be images of $z_2, \cdots, z_r \in R_1$. Then $\overline{z_2}, \cdots, \overline{z_r} \in R_1'$ is a M'-γ- sequence.

So, by the induction hypothesis, we have

$$\operatorname{reg}_{R'} \Omega_{R'}^1 (M') = \max\left\{ \sup\left(k \underset{R'}{\otimes} \Omega_{R'}^1 (M') \right), \operatorname{reg}_{\overline{R'}} \Omega_{\overline{R'}}^1 \left(\overline{M'} \right) \right\} = \max\left\{ \sup\left(k \underset{R}{\otimes} \Omega_{R'}^1 (M') \right), \operatorname{reg}_{\overline{R}} \Omega_{\overline{R}}^1 \left(\overline{M} \right) \right\},$$

where $\overline{R'} := R' \big/ \left(\overline{z_2}, \cdots, \overline{z_r} \right) R' \simeq R \big/ (z_1, \cdots, z_r) R = \overline{R}$, $\overline{M'} := \overline{R'} \underset{R'}{\otimes} M' \simeq \overline{M}$.

Using $\sup\left(k\underset{R}{\otimes}\Omega^1_R M\right)\geq \sup\left(k\underset{R}{\otimes}\Omega^1_{R'}(M')\right)$ by Proposition 4.3, we conclude that

$$\operatorname{reg}_R \Omega^1_R M = \max\left\{\sup\left(k\underset{R}{\otimes}\Omega^1_R M\right), \sup\left(k\underset{R}{\otimes}\Omega^1_{R'}(M')\right), \operatorname{reg}_{\overline{R}}\Omega^1_{\overline{R}}\left(\overline{M}\right)\right\} = \max\left\{\sup\left(k\underset{R}{\otimes}\Omega^1_R M\right), \operatorname{reg}_{\overline{R}}\Omega^1_{\overline{R}}\left(\overline{M}\right)\right\} \quad \square$$

6. In the case of modules whose first syzygies are generated by the elements of the same degree

In this section, we will arrive at our main goal that is Theorem 6.4.

Lemma 6.1. *Assume that* $M\in\mathcal{A}$ *with* $\operatorname{deg-Supp}\left(k\underset{R}{\otimes}\Omega^1_R M\right)=\{d\}$ *for some* $d\in\mathbb{Z}$.

If $\underline{z}=z_1,\cdots,z_r\in R_1\ (r\geq 1)$ *is a* M-γ- *sequence, then the following are equivalent:*

(1) $\operatorname{reg}_R\Omega^1_R M = d$ *;*

(2) $\operatorname{reg}_{\overline{R}}\Omega^1_{\overline{R}}\left(\overline{M}\right)=d$ *or* $\Omega^1_{\overline{R}}\left(\overline{M}\right)=0$*, where* $\overline{R}:= R\big/(z_1,\cdots,z_r)R$ *and* $\overline{M}:= M\big/(z_1,\cdots,z_r)M$.

Proof. $(1)\Rightarrow(2)$: By Proposition 4.3, $\operatorname{deg-Supp}\left(k\underset{R}{\otimes}\Omega^1_{\overline{R}}\overline{M}\right)\subseteq \operatorname{deg-Supp}\left(k\underset{R}{\otimes}\Omega^1_R M\right)=\{d\}$ If $\Omega^1_{\overline{R}}\left(\overline{M}\right)\neq 0$, then $\varnothing\neq \operatorname{deg-Supp}\left(k\underset{R}{\otimes}\Omega^1_{\overline{R}}\overline{M}\right)=\{d\}$. Using Proposition 5.2, we have $d\leq \operatorname{reg}_{\overline{R}}\Omega^1_{\overline{R}}\left(\overline{M}\right)\leq \operatorname{reg}_R\Omega^1_R M = d$. This implies $\operatorname{reg}_{\overline{R}}\Omega^1_{\overline{R}}\left(\overline{M}\right)=d$.

$(2)\Rightarrow(1)$: Using Proposition 5.2, if $\Omega^1_{\overline{R}}\left(\overline{M}\right)=0$, then $\operatorname{reg}_R\Omega^1_R M = \max\left\{\sup\left(k\underset{R}{\otimes}\Omega^1_R M\right)=d, \operatorname{reg}_{\overline{R}}\Omega^1_{\overline{R}}\left(\overline{M}\right)=-\infty\right\}=d$.

Similarly, if $\operatorname{reg}_{\overline{R}}\Omega^1_{\overline{R}}\left(\overline{M}\right)=d$, then $\operatorname{reg}_R\Omega^1_R M = \max\left\{\sup\left(k\underset{R}{\otimes}\Omega^1_R M\right)=d, \operatorname{reg}_{\overline{R}}\Omega^1_{\overline{R}}\left(\overline{M}\right)=d\right\}=d$. $\square$

Lemma 6.2. *Let* $M\in\mathcal{A}$ *with* $\operatorname{deg-Supp}\left(k\underset{R}{\otimes}\Omega^1_R M\right)=\{d\}$ *for some* $d\in\mathbb{Z}$*. Then the following hold:*

If $\operatorname{reg}_R\Omega^1_R M = d$*, then* $\operatorname{deg-Supp}\gamma(M)=\{d-1\}$ *or* $\gamma(M)=0$.

Proof. We recall that $R=k[x_1,\cdots,x_v]$ is a polynomial ring. The following hold:

$$\gamma(M)(-v)\simeq \mathrm{H}_v(x_1,\cdots,x_v;M)\simeq {}^*\mathrm{Tor}^R_v(k, M)\simeq {}^*\mathrm{Tor}^R_{v-1}\left(k, \Omega^1_R M\right),$$

where $\mathrm{H}_v(x_1,\cdots,x_v;M)$ denote the Koszul homology of M with respect to $x_1,\cdots,x_v\in R$.

So, if $\gamma(M)\neq 0$, then we have $\{\nu-1+d\}=\operatorname{deg-Supp}\left({}^{*}\operatorname{Tor}_{\nu-1}^{R}\left(k,\Omega_R^1 M\right)\right)=\operatorname{deg-Supp}\left(\gamma(M)(-\nu)\right)$ since $\Omega_R^1 M$ has d-linear resolution. This implies $\operatorname{deg-Supp}\gamma(M)=\{d-1\}$. □

Proposition 6.3. *Let* $M\in\mathcal{A}$ *with* $\operatorname{deg-Supp}\left(k\underset{R}{\otimes}\Omega_R^1 M\right)=\{d\}$ *for some* $d\in\mathbb{Z}$ *or* $\Omega_R^1 M=0$.

Then the following are equivalent:

(1) $\operatorname{deg-Supp}\gamma(M)=\{d-1\}$ *or* $\gamma(M)=0$;

(2) $\sup\gamma(M)=d-1$ *or* $\gamma(M)=0$;

(3) $\mathfrak{m}^i\Omega_R^1 M=\left(\operatorname{Sat}_{F_M}\Omega_R^1 M\right)_{\geq d+i}$ *for any nonnegative integer* $i\geq 0$, *where we assume that* $\mathfrak{m}^0:=R$;

(4) Any $z\in R_1$ *with* $[z]\in\operatorname{nzd}\left(M/\Gamma_{\mathfrak{m}}(M)\right)$ *is a* $M^{\#*i}$*-*γ*- regular element, especially,* $\gamma\text{-nzd}\left(M^{\#*i}\right)=\operatorname{nzd}\left(M/\Gamma_{\mathfrak{m}}(M)\right)$

for any nonnegative integer $i\geq 0$;

(5) $\gamma\text{-depth}\,M^{\#*i}\geq 1$ *for any nonnegative integer* $i\geq 0$;

(6) $\gamma\text{-depth}\,M\geq 1$.

Proof. $(1)\Rightarrow(2)$, $(5)\Rightarrow(6)$ these implications are trivial. $(4)\Rightarrow(5)$ holds since $\operatorname{nzd}\left(M/\Gamma_{\mathfrak{m}}(M)\right)\neq\varnothing$ by Ref(5).Remark 4.9. $(6)\Rightarrow(1)$ follows from Lemma 3.3, that is; $\operatorname{deg-Supp}\left(\gamma(M)(-1)\right)\subseteq\operatorname{deg-Supp}\left(k\underset{R}{\otimes}\Omega_R^1 M\right)=\{d\}$.

We only prove $(2)\Rightarrow(3)$, $(3)\Rightarrow(4)$.

Let $0\to\Omega_R^1 M\to F_M\underset{\pi_M}{\to}M\to 0$ be a short exact sequence with $\pi_M:F_M\to M$ a minimal grade free cover of M and let $L:=\operatorname{Sat}_{F_M}\Omega_R^1 M\subseteq F_M$ the saturation module of $\Omega_R^1 M$ in F_M. Then $L=\operatorname{Sat}_{F_M}\mathfrak{m}^i\Omega_R^1 M\subseteq F_M$ for any integer $i\geq 0$, we have the following short exact sequence: (A) $0\to\mathfrak{m}^i\Omega_R^1 M\to L\to\Gamma_{\mathfrak{m}}\left(M^{\#*i}\right)\to 0$.

Especially, we have (B) $0\to\Omega_R^1 M\to L\to\Gamma_{\mathfrak{m}}(M)\to 0$.

$(2)\Rightarrow(3)$: First, we prove that $\Omega_R^1 M=L_{\geq d}$.

If $\gamma(M)=0$, then $\Gamma_{\mathfrak{m}}(M)=0$, from the short exact sequence (B), $\Omega_R^1 M=L=L_{\geq d}$ since $\operatorname{deg-Supp}\left(k\underset{R}{\otimes}\Omega_R^1 M\right)=\{d\}$.

If $\gamma(M)\neq 0$, then $\sup\gamma(M)=d-1=\sup\Gamma_{\mathfrak{m}}(M)$, so, $\Gamma_{\mathfrak{m}}(M)_{\geq d}=0$, we have $\Omega_R^1 M=\Omega_R^1 M_{\geq d}=L_{\geq d}$.

We remark that $\Omega_R^1 M_{\langle d\rangle}=\Omega_R^1 M=L_{\geq d}=L_{\langle d\rangle}$ this implies $\mathfrak{m}^i\Omega_R^1 M=\mathfrak{m}^i L_{\langle d\rangle}=L_{\geq d+i}$ for any nonnegative integer $i\geq 0$.

$(3)\Rightarrow(4)$: Using our assumption, $L_{\geq d+i}/L_{\geq d+i+1}\simeq\mathfrak{m}^i\Omega_R^1 M/\mathfrak{m}^{i+1}\Omega_R^1 M$ for $i\geq 0$.

Moreover, from the short exact sequence (A), we have the following short exact sequence:

$$0\to L_{\geq d+i}\to L\to\Gamma_{\mathfrak{m}}\left(M^{\#*i}\right)\to 0.$$

We remark that $\mathbb{P}R_1=\operatorname{nzd}(L)$ since $L\subseteq F_M$. Let $z\in R_1$ with $[z]\in\operatorname{nzd}\left(M/\Gamma_{\mathfrak{m}}(M)\right)\subseteq\mathbb{P}R_1=\operatorname{nzd}(L)$. Then we have

$$\gamma_z\left(\Gamma_{\mathfrak{m}}\left(M^{\#*i}\right)\right)=\left\{\xi\in L \mid z\xi\in L_{\ge d+i}\right\}/L_{\ge d+i}=L_{\ge d+i-1}/L_{\ge d+i}\simeq L_{d+i-1}\simeq \mathfrak{m}^{i-1}\Omega_R^1 M/\mathfrak{m}^i\Omega_R^1 M \quad \text{for} \quad i\ge 1 .$$

From the above isomorphism, we have $\beta_1^R\left(M^{\#*(i-1)}\right)=\operatorname{length}\mathfrak{m}^{i-1}\Omega_R^1 M/\mathfrak{m}^i\Omega_R^1 M=\operatorname{length}\gamma\left(\Gamma_{\mathfrak{m}}\left(M^{\#*i}\right)\right)$. Using Proposition 4.1, we have $[z]\in w\text{-}\gamma\text{-nzd}\left(\Gamma_{\mathfrak{m}}\left(M^{\#*i}\right)\right)$. Hence we have $[z]\in w\text{-}\gamma\text{-nzd}\left(\Gamma_{\mathfrak{m}}\left(M^{\#*i}\right)\right)\cap \operatorname{nzd}\left(M/\Gamma_{\mathfrak{m}}\left(M^{\#*i}\right)\right)=w\text{-}\gamma\text{-nzd}\left(M^{\#*i}\right)$ for $i\ge 1$. □

Theorem 6.4. *Let* $M\in\mathcal{A}$ *with* $\operatorname{deg-Supp}\left(k\otimes_R\Omega_R^1 M\right)=\{d\}$ *for some* $d\in\mathbb{Z}$ *or* $\Omega_R^1 M=0$.

Then the following are equivalent:

(1) $\gamma\text{-depth}\,M\ge \text{Kull-dim}\,R$*;*

(2) $M=0$ *or there exists* $\underline{z}=z_1,\cdots,z_\nu\in R_1$ $(r\ge 1)$ *a* M-γ-*sequence with* $\nu=\text{Kull-dim}\,R$*, especially where* $M\ne 0$.

(3) $\Omega_R^1 M=0$ *or* $\operatorname{reg}_R\Omega_R^1 M=d$ *, especially where* $\Omega_R^1 M\ne 0$.

Proof. (1)$\Rightarrow$(2): This is clear by the definition of γ-depth .

(2)$\Rightarrow$(3): If $\Omega_R^1 M\ne 0$, then especially $M\ne 0$. So, by the assumption, there exists $\underline{z}=z_1,\cdots,z_\nu\in R_1$ $(r\ge 1)$ a M-γ- sequence with $\nu=\text{Kull-dim}\,R$. Let $\overline{R}:=R/(z_1,\cdots,z_\nu)R\simeq k$ and $\overline{M}:=\overline{R}\otimes_R M$ a finite dimensional k -vector space with $\Omega_{\overline{R}}^1\overline{M}=\Omega_k^1\overline{M}=0$.

Using Proposition 5.2, $\operatorname{reg}_R\Omega_R^1 M=\max\left\{\sup\left(k\otimes_R\Omega_R^1 M\right),\operatorname{reg}_{\overline{R}}\Omega_{\overline{R}}^1\left(\overline{M}\right)\right\}=\max\left\{\sup\left(k\otimes_R\Omega_R^1 M\right)=d,\operatorname{reg}_k 0=-\infty\right\}=d$.

(3)$\Rightarrow$(1): If $\Omega_R^1 M=0$ then M is zero or a graded free module over R . So, we have $\gamma\text{-depth}\,M\ge\text{Kull-dim}\,R$.

Let $\operatorname{reg}_R\Omega_R^1 M=d$. We prove this induction on $\text{Kull-dim}\,R=\nu$. If $\nu=1$, automatically $\gamma\text{-depth}\,M=1$.

Let $\nu>1$. Since $\operatorname{reg}_R\Omega_R^1 M=d$, especially $\Omega_R^1 M\ne 0$, we have $\operatorname{deg-Supp}\gamma(M)=\{d-1\}$ by Lemma 6.2. Hence $\gamma\text{-depth}\,M\ge 1$ by Proposition 6.3. So, we can take a M-γ- regular element $0\ne z\in R_1$ and let $\overline{R}:=R/zR$, $\overline{M}:=M/zM$.

It is enough to show that $\gamma\text{-depth}\,\overline{M}=\text{Kull-dim}\,\overline{R}$, since this is equivalent to $\gamma\text{-depth}\,M=\text{Kull-dim}\,R$.

If $\Omega_{\overline{R}}^1\left(\overline{M}\right)=0$, then $\overline{M}$ is a graded free module over $\overline{R}$, so we have $\gamma\text{-depth}\,\overline{M}=\text{Kull-dim}\,\overline{R}$.

If $\Omega_{\overline{R}}^1\left(\overline{M}\right)\ne 0$, then by Lemma 6.1, we have $\operatorname{reg}_{\overline{R}}\Omega_{\overline{R}}^1\left(\overline{M}\right)=d$ and $\text{Kull-dim}\,\overline{R}=\nu-1<\nu$. So applying the induction hypothesis, we have $\gamma\text{-depth}\,\overline{M}=\text{Kull-dim}\,\overline{R}$. □

7. Application to computing Poincare series of graded modules

As an application, we give explicit form of Poincare series of graded modules sated in Ref.(5) Lemma 6.1.

Notation 7.1. Let $M\in\mathcal{A}$. $P_M=P_M^R$ the graded Poincare series and H_M the Hilbert series of M are defined as follows:

(1) $P_M(t,u)=P_M^R(t,u):=\sum_{i\in\mathbb{Z}_{\ge 0},\, j\in\mathbb{Z}}\beta_{ij}^R(M)t^iu^j\in\mathbb{Z}[t,u^{-1}]\ u$,

where we denote $\beta_{ij}^R(M) := \dim_k {}^*\mathrm{Tor}_i^R(k, M)_j$ the graded Betti number of M.

(2) $H_M(u) := \sum_{j\in\mathbb{Z}} \dim_k M_j \in \mathbb{Z}[u^{-1}]\ u$.

Lemma 7.2. *Assume that* $M \in \mathcal{A}$ *and* $0 \neq z \in R_1$ *with* $[z] \in \gamma\text{-nzd}(M)$. *Let denote* $\overline{R} := R/zR$ *and* $\overline{M} := M/zM$. *Then the following equation holds:*

$$P_M^R(t,u) = H_{M\otimes k}(u) + P^{\overline{R}}_{\Omega^1_{\overline{R}}\overline{M}}(t,u)t + P^{\overline{R}}_{\gamma(M)}(t,u)tu .$$

Proof. Let denote $\overline{\Omega^1_R M} := \Omega^1_R M / z\Omega^1_R M$. We remark that z is a regular element on $\Omega^1_R M$. By Proposition 4.3, we have

$$P^R_{\Omega^1_R M}(t,u) = P^{\overline{R}}_{\overline{\Omega^1_R M}}(t,u) = P^{\overline{R}}_{\Omega_{\overline{R}}\overline{M}}(t,u) + P^{\overline{R}}_{\gamma(M)}(t,u)u .$$

On the other hand, the following is always holds:

$$P_M^R(t,u) = H_{M\otimes k}(u) + P^R_{\Omega^1_R M}(t,u)t .$$

Hence the assertion follows. □

Using the same notations in Ref.(5) Lemma 6.1, we have the following Proposition:

Proposition 7.3.

(1) *Let* $M := \mathfrak{R}_\lambda(e)$ $(\lambda \in k, e \geq 1)$ or $\mathfrak{R}_\infty(e)$. *Then we have* $P_M(t,u) = e\left(1+u(1+u)t+u^3t^2\right)$.

(2) *Let* $M := \mathfrak{P}(d)$ $(d \geq 0)$. *Then we have* $P_M(t,u) = d + \left((d-1)+(d+2)u\right)ut + (d+1)u^3t^2$.

Proof. Applying Lemma 7.2, we have the following:

(1) $P_M^R(t,u) = e + eut + eu(1+tu)tu = e\left(1+u(1+u)t+u^3t^2\right)$.

Since, using Ref(5). Proposition 6.2, we can easily see that the following hold:

$$H_{M\otimes k}(u) = H_{k^e}(u) = e\ ,\quad P^{\overline{R}}_{\Omega^1_{\overline{R}}\overline{M}}(t,u) = P^{\overline{R}}_{\left(\overline{R}(-1)\right)^e}(t,u) = eu \quad \text{and} \quad P^{\overline{R}}_{\gamma(M)}(t,u) = P^{\overline{R}}_{k^e(-1)}(t,u) = eu(1+tu) .$$

(2) $P_M^R(t,u) = d + \left((d-1)u+u^2\right)t + (d+1)u(1+tu)tu = d + \left((d-1)+(d+2)u\right)ut + (d+1)u^3t^2$.

Similarly, since we have the following:

$$H_{M\otimes k}(u) = H_{k^d}(u) = d\ ,\quad P^{\overline{R}}_{\Omega^1_{\overline{R}}\overline{M}}(t,u) = P^{\overline{R}}_{\left(\overline{R}(-1)\right)^{d-1}\oplus\overline{R}(-2)}(t,u) = (d-1)u+u^2 \quad \text{and} \quad P^{\overline{R}}_{\gamma(M)}(t,u) = P^{\overline{R}}_{k^{d+1}(-1)}(t,u) = (d+1)u(1+tu)$$

by Ref(5). Proposition 6.2. □

Remark 7.4. We remark that $H_M(u) = P_M^R(-1,u)H_R(u)$ holds in general. If $R = k[x,y]$, then $H_R(u) = \dfrac{1}{(1-u)^2}$.

With the same notations as Proposition 7.3 (1) and (2) respectively, in fact, we can confirm the following equations:

(1) $H_M(u)=\dfrac{e\left(1-u\left(1+u\right)+u^3\right)}{\left(1-u\right)^2}=e\left(1+u\right)$.

(2) $H_M(u)=\dfrac{d-\left(\left(d-1\right)+\left(d+2\right)u\right)u+\left(d+1\right)u^3}{\left(1-u\right)^2}=d+\left(d+1\right)u$.

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