Paper

Almost regular sequences on graded modules associated with a subfunctor of the 0-th local cohomology functor I

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For graded modules whose first syzygies are generated by elements of the same degree, to have a component-wise linear syzygy so actually to have linear syzygy is equivalent to that there exists a γ- regular sequence whose length is equal to the Kull dimension of the base ring. This is the main result of this two-part series of papers, which is also a partial extension of the result of Watanabe and Harima, see Ref(1). In the first part of papers, here we give some preparatory results, especially, we introduce the weak γ- regular sequence in the category of finitely generated graded modules over a polynomial ring whose coefficient field is an infinite field and give some fundamental results concerning the weak γ- regular sequence.

Keywords: Almost regular sequences, weak γ- regular elements, weak γ- regular sequences, weak γ- depth, graded modules

1. Introduction

In the first part of this two-part series of papers, we introduce '*weak γ -regular elements, weak γ - regular sequences and weak γ - depth's*' on graded modules, which are defined by a subfunctor γ of the 0-th local cohomology functor $\Gamma_{\mathfrak{m}} = H^0_{\mathfrak{m}}$. All weak γ -regular elements are almost regular elements which is the terminology sited in Herzog's book, see Ref.(2). In this sense, a weak γ -regular element is a generalization of an ordinary regular element. But the behaviors of weak γ -regular elements on short exact sequences are difficult to control unlikely in the case of regular elements. So, we strengthen the notion weak γ -regular to γ -regular by adding a slight condition in the second part of this series of papers, then γ -regular sequences have good properties among almost regular sequences.

In fact, for a nonzero principal graded module, its first syzygy ideal being a $\mathfrak{m}$ -full ideal, see Ref.(3), is equivalent to that the principal module has a γ -regular element in our terminology. In Ref.(1), they proved surprising result that completely $\mathfrak{m}$ -full ideals are just the ideals having component-wise linear resolutions, also see Ref.(4) or Ref.(5). The first syzygy ideal of a nonzero principal graded module is a completely $\mathfrak{m}$ -full ideal if and only if the γ -depth of the principal graded module is equal to the Kull dimension of base polynomial ring.

To extend this result to arbitrary modules is our main motivation, in spite not completely achieving this goal yet. So, in the first part of this series of papers, we give several fundamental results on weak-γ -regular elements, weak-γ -sequences and weak-γ -depth toward solving this problem. The notations γ -regular elements, γ -sequences and γ -depth will appear in the second part.

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2. Preliminary

Let $R:=k[x_1,\cdots,x_\nu]=\underset{i\geq 0}{\oplus}R_i$ be a polynomial ring in $\nu\geq 1$ variables $x_1,\cdots,x_\nu$ over an infinite field k and let $\mathfrak{m}:=\bigoplus_{i>0}R_i$ be the graded maximal ideal of R, we sometimes use the notation $\mathfrak{m}_i:=R_i$ for any positive integer $i>0$. We always denote the set of integers by $\mathbb{Z}$. We denote $\mathrm{grMod}\,R$ the category of graded modules over R, whose graded components are all finite dimensional over k, that is; for any object $M=\underset{i\in\mathbb{Z}}{\oplus}M_i$ in $\mathrm{grMod}\,R$, $\dim_k M_i<\infty$ for all $i\in\mathbb{Z}$, where M_i the graded component of degree i of M and $M(j)\in\mathrm{grMod}\,R$ the j-shift of M in $\mathrm{grMod}\,R$, where $M(j)_i:=M_{i+j}$ for any $i,j\in\mathbb{Z}$. We remark that homomorphisms in $\mathrm{grMod}\,R$ are morphisms of degree 0, that is; R-linear maps which preserve degrees;

$$\mathrm{Hom}_{\mathrm{grMod}\,R}(M,N)=\left\{f\in\mathrm{Hom}_R(M,N)\,\middle|\,f(M_i)\subseteq N_i \text{ for any } i\in\mathbb{Z}\right\}.$$

We also denote the category of finitely generated graded modules over R by $\mathcal{A}:=\mathrm{grmod}\,R$ as a full subcategory of $\mathrm{grMod}\,R$. $\mathcal{A}$ has $\mathrm{hom}_{\mathcal{A}}$ the inner hom functor, that is; for any $M,N\in\mathcal{A}$:

$$\mathrm{hom}_{\mathcal{A}}(M,N):=\underset{i\in\mathbb{Z}}{\oplus}\mathrm{hom}_{\mathcal{A}}(M,N)_i\in\mathcal{A},$$

where $\mathrm{hom}_{\mathcal{A}}(M,N)_i=\mathrm{Hom}_{\mathcal{A}}(M,N(i))$ for $i\in\mathbb{Z}$. We remark in fact that $\mathrm{hom}_{\mathcal{A}}(M,N)=\mathrm{Hom}_R(M,N)$ for any $M,N\in\mathcal{A}$, since $\mathrm{hom}_{\mathcal{A}}(R,N)_i\simeq N_i$ and $\mathrm{hom}_{\mathcal{A}}(R,N)\simeq N\simeq\mathrm{Hom}_R(M,N)$, so let $F_1\to F_0\to M\to 0\,(\text{exact})$ be a free representation in $\mathcal{A}$, we have the following commutative diagram with exact rows and vertical morphisms being isomorphisms:

$$\begin{array}{ccccccc} 0 & \to & \mathrm{hom}_{\mathcal{A}}(M,N) & \to & \mathrm{hom}_{\mathcal{A}}(M,F_0) & \to & \mathrm{hom}_{\mathcal{A}}(M,F_1) \\ & & & & \downarrow & & \downarrow \\ 0 & \to & \mathrm{Hom}_R(M,N) & \to & \mathrm{Hom}_R(M,F_0) & \to & \mathrm{Hom}_R(M,F_1) \end{array}.$$

From now on, we define $\mathrm{Hom}_R(M,N)_i:=\mathrm{Hom}_{\mathcal{A}}(M,N(i))$, then we can see that $\mathrm{Hom}_R(M,N)=\underset{i\in\mathbb{Z}}{\oplus}\mathrm{Hom}_R(M,N)_i\in\mathcal{A}$ for any $M,N\in\mathcal{A}$. We mainly work in the category $\mathcal{A}$, however, remark that we sometimes drop denoting the shift functor (i) for simplicity.

3. Definition of weak γ-regular elements and sequences

For a nonzero proper homogeneous ideal $0\neq I\subseteq R$ we denote:

$$\gamma_I(M):=\left\{\xi\in M\,\middle|\,I\xi=0\right\},$$

then, we can see that $\gamma_I(M)\simeq\mathrm{Hom}_R(R/I,M)$ through the natural isomorphism. For a principal homogeneous ideal $I=Rf$ with $0\neq f\in R$ a homogeneous element, we denote $\gamma_I(M)=\gamma_f(M)$ for simplicity.

Moreover, if $I=\mathfrak{m}$, we simply denote $\gamma(M)=\gamma_{\mathfrak{m}}(M)$ which is so-called 'socle of M', then γ can be seen a left exact subfunctor of 0-th local cohomology functor $\Gamma_{\mathfrak{m}}=H^0_{\mathfrak{m}}$, where $\Gamma_{\mathfrak{m}}(M)=H^0_{\mathfrak{m}}(M)=\sum_{i\geq 1}\gamma_{\mathfrak{m}^i}(M)$.

Notation 3.1. Let $0\neq I\subseteq R$ be a nonzero proper homogeneous ideal. We denote the i-th derived functor of γ_I by γ_I^i. More precisely, since $\gamma_I(M)\simeq\mathrm{Hom}_R(R/I,M)$, $\gamma_I^i(M)\simeq{}^*\mathrm{Ext}_R^i(R/I,M)$ the i-th graded extension module for an integer $i\geq 0$, where we assume that $\gamma_I^0 := \gamma_I$. If $I = \mathfrak{m}$, then we denote $\gamma_{\mathfrak{m}}^i = \gamma^i$ and if $I = Rf$, then denote $\gamma_{Rf}^i = \gamma_f^i$ for simplicity.

Remark 3.2. Since $\gamma_I^i(M) \simeq {}^*\mathrm{Ext}_R^i(R/I, M)$, we remark that if $0 \neq I \subseteq J \subseteq R$ nonzero proper homogeneous ideals, then there is the natural transformation $\mathrm{Ext}_R^i(\varphi, \_) : \gamma_J^i \to \gamma_I^i$ induced by the projection $\varphi : R/I \to R/J$.

Notation 3.3. Let $0 \neq z \in R_1, M \in \mathcal{A}$. We denote $\times z : M \to M \in \mathrm{Hom}_R(M, M)_1$ the degree one morphism multiplying by z.

Definition 3.4. Let $0 \neq z \in R_1, M \in \mathcal{A}$. We call that z is a ***weak*** γ ***-regular element*** on M, a ***weak*** M-γ- ***regular element*** or a ***weak*** γ- ***'non zero-divisor'*** on M if the following condition holds:

$$0 \to \gamma(M) \to M \underset{\times z}{\to} M \ (\text{exact}), \text{ that is; } \gamma(M) = \gamma_z(M).$$

Remark 3.5. The following holds:

(1) By definition, if $M = 0$, then any $0 \neq z \in R_1$ is a weak γ -regular element on M.

(2) If $\mathfrak{m}M = 0$, that is; $\gamma(M) = M$, then any $0 \neq z \in R_1 = \mathfrak{m}_1$ is a weak γ -regular element on M.

(3) We remark that weak γ -regular elements on M are all contained in $R_1 \setminus \{0\}$ if they exist.

(4) $\gamma(M) = \gamma_z(M)$ if and only if $\mathfrak{m}\gamma_z(M) = 0$.

(5) Since $\gamma_z(M)(-1) \simeq {}^*\mathrm{Tor}_1^R\left(R/zR, M\right)$, $\gamma(M) = \gamma_z(M)$ is equivalent to $\mathfrak{m}\,{}^*\mathrm{Tor}_1^R\left(R/zR, M\right) = 0$,

where ${}^*\mathrm{Tor}_1^R(R/zR, \_)$ denote the graded torsion functor.

Definition 3.6. A sequence $\underline{z} = z_1, \cdots, z_r \ (r \geq 1)$ of elements in R_1 is called a ***weak*** γ ***-regular sequence*** on M, more simply, a ***weak*** M-γ- ***sequence*** of length r if the following conditions are satisfied:

(1) The set $\{z_1, \cdots, z_r\}$ is linearly independent over the field k.

(2) First, z_1 is a weak γ -regular element on M.

If $r \geq 2$, then z_i is a weak γ -regular element on $M/(z_1, \cdots, z_{i-1})M$ for each $i = 2, \cdots, r$.

Remark 3.7. Two conditions (1) and (2) in the above definition are equivalent to the following condition (3):

$$(3)\ \overline{z_i}^{(i-1)} \in \left(\overline{R}^{(i-1)}\right)_1 \text{ is a weak } \gamma \text{-regular elements on } \overline{M}^{(i-1)} \text{ for } i = 1, \cdots, r,$$

where $\overline{R}^{(0)} := R$, $\overline{R}^{(i)} := R/(z_1, \cdots, z_i)R$, $\overline{M}^{(i)} := M \otimes_R \overline{R}^{(i)}$ and $\overline{z_i}^{(i-1)}$ denoting the image of z_i in $\overline{R}^{(i-1)}$ for $i = 1, \cdots, r$.

Remark 3.8. Different form the ordinary M- regular sequence, see Definition 1.1.1 in Ref.(6), a weak M-γ- sequence depends on its order, see Example 3.10. Moreover we do not know whether all maximal weak M-γ- sequences have the same length or not by lacking of universal tool for measuring the length of a maximal weak M-γ- sequence like the local cohomology in the case of ordinary M- regular sequences.

Notation 3.9. Let $M \in \mathcal{A}$. The invariant w-γ-depth$M$ called the ***weak* γ*- depth*** of M is defined as follows:

(1) If $M = 0$, then we define w-γ-depth$M = w$-γ-depth$0 := \infty$ in convention, where we consider that $\mathbb{Z} \cup \{\infty\}$ is an ordered set with $\infty > i$ for all $i \in \mathbb{Z}$ and $\mathbb{Z}$ having ordinary order.

(2) We denote w-γ-depth$M = 0$, if and only if there is no weak γ- regular element on M.

(3) If there is a weak γ- regular element on M and $M \neq 0$, then we define:

$$w\text{-}\gamma\text{-depth}M := \max\{r \mid \underline{z} = z_1, \cdots, z_r \text{ is a weak } M\text{-}\gamma\text{-sequence}\}.$$

Remark 3.10. If $M \neq 0$, then w-γ-depth$M \leq \nu$, where $\nu = \text{Kull-dim}R$ which is equal to the number of variables of R.

Example 3.11. Let $R = k[x,y]$, $\mathfrak{m} = (x,y)R$ and $M = \mathfrak{m}/(\mathfrak{m}^3 + y^2R)$. Then x, y is a M-γ- sequence but y, x is not a M-γ-sequence. Since $\gamma(M) = \gamma_x(M) = \mathfrak{m}^2/(\mathfrak{m}^3 + y^2R)$, $M/xM \simeq \mathfrak{m}/\mathfrak{m}^2 \simeq kx \oplus ky$ but $\gamma_y(M) = (\mathfrak{m}^2 + yR)/(\mathfrak{m}^3 + y^2R) \neq \gamma(M)$.

4. Non zero-divisor loci

Let $0 \neq M \in \mathcal{A}$ and let $\text{Ass}_R M$ be a set of associated prime ideals of M, see Ref.(6) for the definition. Since M is a Noetherian graded module, all associated primes are homogeneous, see Lemma1.5.6 in Ref.(6), especially, $\mathfrak{p} \subseteq \mathfrak{m}$ for all $\mathfrak{p} \in \text{Ass}_R M$ and $\text{Ass}_R M$ is a finite set. Moreover, it is well known that $\text{Ker}(\times f : M \to M) = 0$ if and only if $f \notin \bigcup_{\mathfrak{p} \in \text{Ass}_R M} \mathfrak{p}$ for $0 \neq f \in R$.

Notation 4.1. Let V be a finite dimensional vector space over k, we denote the projective space associated with V by $\mathbb{P}V$ and for any $0 \neq v \in V$, we denote the image of v in $\mathbb{P}V$ by $[v] \in \mathbb{P}V$.

Definition 4.2. Let $M \in \mathcal{A}$. We define $\text{nzd}(M)$ the *non zero-divisor locus* of M in $\mathbb{P}R_1$ as follows:

$$\text{nzd}(M) := \{[z] \in \mathbb{P}R_1 \mid 0 \to M \underset{\times z}{\to} M \ (\text{exact})\}.$$

Remark 4.3. If $M = 0$, then for any $0 \neq z \in R_1$ the following sequence is exact, that is;

$$0 \to M \underset{\times z}{\to} M \ (\text{exact}), \text{ so } \text{nzd}(M) = \text{nzd}(0) = \mathbb{P}R_1.$$

Lemma 4.4. *Let* $0 \neq M \in \mathcal{A}$. *If* $\mathfrak{m} \notin \text{Ass}_R M$, *then* $\text{nzd}(M)$ *is a nonempty Zariski open subset of* $\mathbb{P}R_1$.

Proof. We put $\text{NZD}_1(M) := \{0 \neq z \in R_1 \mid 0 \to M \underset{\times z}{\to} M \ (\text{exact})\}$. Then $\mathfrak{p} \cap R_1 \subsetneq R_1$ is a proper subset for any $\mathfrak{p} \in \text{Ass}_R M$ by the assumption $\mathfrak{m} \notin \text{Ass}_R M$, so $\emptyset \neq \text{NZD}_1(M) = R_1 \setminus \bigcup_{\mathfrak{p} \in \text{Ass}_R M} (\mathfrak{p} \cap R_1)$ is a nonempty Zariski open subset of R_1, since the vector space

R_1 over the infinite field k cannot be covered by a union of finitely many proper subspaces, moreover, which are in general a Zariski closed subset of R_1. Let $\emptyset \neq \mathrm{NZD}_1(M) \underset{\iota}{\rightarrow} R_1 \underset{\pi}{\rightarrow} \mathbb{P}R_1$ be the composition of natural inclusion ι and the natural projection π. We see that $\mathrm{nzd}(M) = \pi \circ \iota(\mathrm{NZD}_1(M)) \subseteq \mathbb{P}R_1$ is a nonempty Zariski open subset. □

Remark 4.5. From the above Lemma 4.4, we can see that $\mathrm{nzd}(M/\Gamma_{\mathfrak{m}}(M))$ is always a nonempty Zariski open subset of $\mathbb{P}R_1$, since $M/\Gamma_{\mathfrak{m}}(M) = 0$ or otherwise $\mathfrak{m} \notin \mathrm{Ass}_R(M/\Gamma_{\mathfrak{m}}(M))$.

Again let $0 \neq M \in \mathcal{A}$. If $\dim_k M < \infty$, then a homothety $\times z$ is a linear map on finite vector space on M, so we can define the rank of a homothety as follows: $\mathrm{rank}(\times z) := \dim_k \mathrm{Im}(\times z) < \infty$. For any positive integer $i > 0$, the set

$$U_{\mathrm{rank} \geq i}(M) := \{[z] \in \mathbb{P}R_1 \mid \mathrm{rank}(\times z : M \to M) \geq i\} \subseteq \mathbb{P}R_1$$

is a Zariski open subset of $\mathbb{P}R_1$, see Ref.(7). Let $r_{\max} := \max\{i \in \mathbb{Z}_{>0} \mid U_{\mathrm{rank}\geq i}(M) \neq \emptyset\}$ and denote $U(M) := U_{\mathrm{rank} \geq r_{\max}}(M)$.

Definition 4.6. Let $M \in \mathcal{A}$. We define $w\text{-}\gamma\text{-}\mathrm{nzd}(M)$ the w-γ -*'non zero-divisor' locus* of M in $\mathbb{P}R_1$ as follows:

$$w\text{-}\gamma\text{-}\mathrm{nzd}(M) := \{[z] \in \mathbb{P}R_1 \mid z \text{ is a weak } \gamma\text{-regular element on } M\}.$$

Lemma 4.7. *Let $0 \neq z \in R_1, M \in \mathcal{A}$. The following conditions are equivalent:*

(1) z is a weak γ- regular element on M ;

(2) The following two sequences are exact:

$$0 \to \gamma(M) \to \Gamma_{\mathfrak{m}}(M) \underset{\times z}{\rightarrow} \Gamma_{\mathfrak{m}}(M)\ (exact) \quad and \quad 0 \to M/\Gamma_{\mathfrak{m}}(M) \underset{\times z}{\rightarrow} M/\Gamma_{\mathfrak{m}}(M)\ (exact).$$

Proof. First, we remark that $\gamma(M) = \gamma(\Gamma_{\mathfrak{m}}(M))$ since $\gamma(M) \subseteq \Gamma_{\mathfrak{m}}(M) \subseteq M$. If $M = \Gamma_{\mathfrak{m}}(M)$, then we have nothing to prove. So, we assume that $M \neq \Gamma_{\mathfrak{m}}(M)$. Then $\mathfrak{m} \notin \mathrm{Ass}_R(M/\Gamma_{\mathfrak{m}}(M))$, in this case, we notice that $M/\Gamma_{\mathfrak{m}}(M)$ contains no nonzero finite length submodule, since any nonzero finite length module has a nonzero socle.

We have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & \mathrm{Ker}\alpha & \underset{\iota}{\rightarrow} & \mathrm{Ker}\beta & \rightarrow & \mathrm{Ker}\rho & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Gamma_{\mathfrak{m}}(M) & \rightarrow & M & \rightarrow & M/\Gamma_{\mathfrak{m}}(M) & \rightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha:=\times z} & & \downarrow{\scriptstyle \beta:=\times z} & & \downarrow{\scriptstyle \rho:=\times z} & & \\
0 & \rightarrow & \Gamma_{\mathfrak{m}}(M) & \rightarrow & M & \rightarrow & M/\Gamma_{\mathfrak{m}}(M) & \rightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & \Gamma_{\mathfrak{m}}(M)/z\Gamma_{\mathfrak{m}}(M) & & & & & &
\end{array},$$

From the above commutative diagram, using the Snake-Lemma, if (2) holds, then $\mathrm{Ker}\rho = 0$ and $\mathrm{Ker}\alpha = \gamma(M) = \mathrm{Ker}\beta$, so (1) holds.

On the other hand, if (1) holds, from the above commutative diagram, $\mathrm{Ker}\beta = \gamma(M)$ and we have, again using the Snake-Lemma,

$$\dim_k \mathrm{Ker}\rho \le \dim_k \mathrm{Ker}\beta + \dim_k \Gamma_{\mathfrak{m}}(M)/z\Gamma_{\mathfrak{m}}(M) = \dim_k \gamma(M) + \dim_k \Gamma_{\mathfrak{m}}(M)/z\Gamma_{\mathfrak{m}}(M) < \infty .$$

Hence $\mathrm{Ker}\rho = 0$ since $\mathrm{Ker}\rho$ is a finite length submodule of $M/\Gamma_{\mathfrak{m}}(M)$, where $\mathfrak{m} \notin \mathrm{Ass}_R(M/\Gamma_{\mathfrak{m}}(M))$, so (2) holds. □

As an immediate consequence of the above Lemma 4.7, we have the following proposition.

Proposition 4.8. *Let* $M \in \mathcal{A}$ *.The following hold:*

$$w\text{-}\gamma\text{-nzd}(M) = w\text{-}\gamma\text{-nzd}(\Gamma_{\mathfrak{m}}(M)) \cap \mathrm{nzd}(M/\Gamma_{\mathfrak{m}}(M)) .$$

Remark 4.9. We remark that $\mathrm{nzd}(M/\Gamma_{\mathfrak{m}}(M)) \ne \varnothing$ by definition. Hence $w\text{-}\gamma\text{-nzd}(M) \ne \varnothing$ if and only if $w\text{-}\gamma\text{-nzd}(\Gamma_{\mathfrak{m}}(M)) \ne \varnothing$.

5. Properties of weak γ- regular elements

In this section, we observe that the properties of weak γ- regular elements.

Lemma 5.1. *Let* $0 \to L \underset{f}{\to} M \underset{g}{\to} N \to 0$ *be a short exact sequence in* $\mathcal{A}$.

If $0 \to \gamma(L) \underset{\gamma(f)}{\to} \gamma(M) \underset{\gamma(g)}{\to} \gamma(N) \to 0$ *is exact, then* $w\text{-}\gamma\text{-nzd}(L) \cap w\text{-}\gamma\text{-nzd}(N) \subseteq w\text{-}\gamma\text{-nzd}(M)$.

Especially, if $w\text{-}\gamma\text{-depth}(L), w\text{-}\gamma\text{-depth}(N) \ge 1$*, then* $w\text{-}\gamma\text{-depth}(M) \ge 1$.

Proof. If $[z] \in w\text{-}\gamma\text{-nzd}(L) \cap w\text{-}\gamma\text{-nzd}(N)$, then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} (0 & \to) & \gamma(L) & \underset{\gamma(f)}{\to} & \gamma(M) & \underset{\gamma(g)}{\to} & \gamma(N) & \to & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \to & \gamma_z(L) & \underset{\gamma_z(f)}{\to} & \gamma_z(M) & \underset{\gamma_z(g)}{\to} & \gamma_z(N) & & \end{array}$$

By the Five-Lemma, the middle vertical inclusion is an isomorphism. Hence $[z] \in w\text{-}\gamma\text{-nzd}(M)$.

Especially, if $w\text{-}\gamma\text{-depth}(L), w\text{-}\gamma\text{-depth}(N) \ge 1$, then $w\text{-}\gamma\text{-nzd}(L)$ and $w\text{-}\gamma\text{-nzd}(N)$ are nonempty Zariski open subset in the irreducible space $\mathbb{P}R_1$, so $w\text{-}\gamma\text{-nzd}(L) \cap w\text{-}\gamma\text{-nzd}(N)$ is also a nonempty Zariski open subset in $\mathbb{P}R_1$. By the above result, we have:

$$\varnothing \ne w\text{-}\gamma\text{-nzd}(L) \cap w\text{-}\gamma\text{-nzd}(N) \subseteq w\text{-}\gamma\text{-nzd}(M)$$

Therefore $w\text{-}\gamma\text{-depth}(M) \ge 1$. □

Lemma 5.2. *Let* $L \subseteq M$ *in* $\mathcal{A}$ *. Then* $\gamma\text{-nzd}(M) \subseteq \gamma\text{-nzd}(L)$*. Especially, if* $\gamma\text{-depth}(M) \ge 1$*, then* $\gamma\text{-depth}(L) \ge 1$.

Proof. If $[z] \in w\text{-}\gamma\text{-nzd}(M)$, then $\gamma(M) = \gamma_z(M)$ by definition, so we have $\gamma(L) = L \cap \gamma(M) = L \cap \gamma_z(M) = \gamma_z(L)$ and $[z] \in w\text{-}\gamma\text{-nzd}(L)$. Hence $w\text{-}\gamma\text{-nzd}(M) \subseteq w\text{-}\gamma\text{-nzd}(L)$ holds.

Especially if $\gamma\text{-depth}(M) \ge 1$, then $\varnothing \ne w\text{-}\gamma\text{-nzd}(M) \subseteq w\text{-}\gamma\text{-nzd}(L)$ so we have $w\text{-}\gamma\text{-depth}(L) \ge 1$. □

Lemma 5.3. *Let* $0 \to L \underset{f}{\to} M \underset{g}{\to} N \to 0$ *be a short exact sequence in* $\mathcal{A}$.

If $\gamma(L) \underset{\gamma(f)}{\simeq} \gamma(M)$ *and* $\mathfrak{m}\gamma_z^1(L)=0$ *with* $[z]\in\gamma\text{-nzd}(L)\cap\gamma\text{-nzd}(M)$, *then* $[z]\in\gamma\text{-nzd}(N)$.

Especially, if $w\text{-}\gamma\text{-depth}(L), w\text{-}\gamma\text{-depth}(M)\geq 1$ *and* $\mathfrak{m}\gamma_z^1(L)=0$, *then* $w\text{-}\gamma\text{-depth}(N)\geq 1$.

Proof. If $[z]\in\gamma\text{-nzd}(L)\cap\gamma\text{-nzd}(N)$, then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \gamma(L) & \underset{\gamma(f)}{\rightarrow} & \gamma(M) & \underset{\gamma(g)}{\rightarrow} & \gamma(N) & \rightarrow & \gamma^1(L) \\ \| & & \| & & \downarrow & & \downarrow \\ \gamma_z(L) & \underset{\gamma_z(f)}{\rightarrow} & \gamma_z(M) & \underset{\gamma_z(g)}{\rightarrow} & \gamma_z(N) & \underset{\delta}{\rightarrow} & \gamma_z^1(L) \end{array}$$

From the above commutative diagram, we see that δ is injective since $\gamma(L) \underset{\gamma(f)}{\simeq} \gamma(M)$, so $\gamma_z(f):\gamma_z(L)\rightarrow\gamma_z(M)$ is also an isomorphism. Hence we have $\mathfrak{m}\gamma_z(N)\subseteq\mathfrak{m}\gamma_z^1(L)=0$. By Remark 3.5.(4), we see that $[z]\in\gamma\text{-nzd}(N)$. □

6. Examples of w-γ- non zero-divisor locus

Let $\text{Kull-dim}R=2$, that is; R is the polynomial ring in two variables $R=k[x,y]$. In this section, we observe that the w-γ- non zero-divisor locus of indecomposable graded modules whose components vanish other than degree zero and one.

Using the classification of the representations of 2-Kronecker quiver, see Ref. (8), we have the following lemma:

Lemma 6.1. *Let* $0\neq M=M_0\oplus M_1\in\mathcal{A}$ *be a nonzero indecomposable graded module and let denote* $\mathfrak{m}^0:=R$. *Then* M *is isomorphic to one of the following three types of modules:*

(1) $\mathfrak{R}_\lambda(e):=\left(\mathfrak{m}^{e-1}\Big/\left(\mathfrak{m}^{e+1}+(\lambda x-y)^e R\right)\right)(e-1)\quad(\lambda\in k,\, e\geq 1),\quad \mathfrak{R}_\infty(e):=\left(\mathfrak{m}^{e-1}\Big/\left(\mathfrak{m}^{e+1}+x^e R\right)\right)(e-1)\quad(e\geq 1)$;

(2) $\mathfrak{P}(d):=\begin{cases}\left(\mathfrak{m}^{d-1}\big/\mathfrak{m}^{d+1}\right)(d-1) & (\text{if } d\geq 1)\\ k(-1) & (\text{if } d=0)\end{cases}$;

(3) $\mathfrak{I}(d):=\begin{cases}\left(\mathfrak{m}^{d-1}\big/\mathfrak{m}^{d+1}\right)^\vee(-d) & (\text{if } d\geq 1)\\ k & (\text{if } d=0)\end{cases}$.

From Lemma 3.2.2. and Lemma 3.3.3 in Ref.(8), we have the following lemma:

Proposition 6.2. *Under the same notations as above, we have the following:*

(1) $w\text{-}\gamma\text{-nzd}(\mathfrak{R}_\lambda(e))=\mathbb{P}R_1\setminus\{[\lambda x-y]\}$ *and* $w\text{-}\gamma\text{-nzd}(\mathfrak{R}_\infty(e))=\mathbb{P}R_1\setminus\{[x]\}$ *for any* $\lambda\in k$, $e\geq 1$.

Let $M:=\mathfrak{R}_\lambda(e)$ $(\lambda\in k,\, e\geq 1)$ or $\mathfrak{R}_\infty(e)$ *and* $0\neq z\in R_1$ *with* $[z]\in w\text{-}\gamma\text{-nzd}(M)$.

Then we have $\gamma(M)\simeq k^e(-1)$, $M/zM\simeq k^e$ *and* $w\text{-}\gamma\text{-depth}\,M=2$.

(2) $w\text{-}\gamma\text{-nzd}(\mathfrak{P}(d))=w\text{-}\gamma\text{-nzd}(\mathfrak{I}(0))=\mathbb{P}R_1$ *for any* $d\geq 0$, *especially* $w\text{-}\gamma\text{-depth}\,\mathfrak{I}(0)=2$.

Let $0 \neq z \in R_1$ *with* $[z] \in w\text{-}\gamma\text{-nzd}(\mathfrak{P}(d))$ $(d \geq 0)$.

Then we have $\gamma(\mathfrak{P}(d)) \simeq k^{d+1}(-1)$, $\mathfrak{P}(d)/z\mathfrak{P}(d) \simeq k^{d-1} \oplus k[t]/t^2k[t]$ *and* $w\text{-}\gamma\text{-depth}\,\mathfrak{P}(d) = 2$ *for any* $d \geq 0$.

(3) $w\text{-}\gamma\text{-nzd}(\mathfrak{I}(d)) = \varnothing$, *especially* $w\text{-}\gamma\text{-depth}\,\mathfrak{I}(d) = 0$, *for any* $d \geq 1$.

Remark 6.3. In the second part of this series of papers, we introduce the γ- non zero-divisor locus $\gamma\text{-nzd}(M)$ and the γ- depth $\gamma\text{-depth}\,M$ of graded module $M \in \mathcal{A}$. By the definition, we can easily see that the following hold:

(1) $\gamma\text{-nzd}(\mathfrak{R}_\lambda(e)) = \mathbb{P}R_1 \setminus \{[\lambda x - y]\}$, especially $\gamma\text{-depth}\,\mathfrak{R}_\lambda(e) = 2$, for any $\lambda \in k, e \geq 1$ and $\gamma\text{-nzd}(\mathfrak{R}_\infty(e)) = \mathbb{P}R_1 \setminus \{[x]\}$, especially $\gamma\text{-depth}\,\mathfrak{R}_\infty(e) = 2$.

(2) $\gamma\text{-nzd}(\mathfrak{P}(d)) = \gamma\text{-nzd}(\mathfrak{I}(0)) = \mathbb{P}R_1$, especially $\gamma\text{-depth}\,\mathfrak{P}(d) = \gamma\text{-depth}\,\mathfrak{I}(0) = 2$, for any $d \geq 0$.

(3) $\gamma\text{-nzd}(\mathfrak{I}(d)) = \varnothing$, especially $\gamma\text{-depth}\,\mathfrak{I}(d) = 0$, for any $d \geq 1$.

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References

(1) T. Harima and J. Watanabe: "*Completely* $\mathfrak{m}$ *-full ideals and componentwise linear ideals*", Math. Proc. Cambridge Philos. Soc. **158**, pp. 239-248(2015).
(2) J. Herzog and T. Hibi: "*Monomial Ideals*", Graduate Texts in Mathematics **260**, Springer (2010).
(3) J. Watanabe: "*m-Full Ideals*", Nagoya Math. Journal, **106**, pp. 101-111 (1987).
(4) L. L. Avramov and D. Eisenbud: "*Regularity of modules over a Koszul algebra*", J. Algebra. **153**(1), pp. 85-90 (1992).
(5) D. Lu and D. Zhou: "*Componentwise linear modules over a Koszul algebra*", Taiwanese Journal of Mathematics. **17**(6), pp. 2135-2147 (2013).
(6) W. Bruns and J. Herzog: *Cohen-Macauly rings Revised Edition*, Cambridge studies in advanced mathematics **39**, Cambridge University Press (1997).
(7) S. Isogawa : "*Rees property and its related properties of modules*", Research Reports of NIT, Kumamoto College. **6**, pp. 74-81(2014).
(8) S. Isogawa, "*Weak Lefschetz properties of graded modules over a polynomial ring in two variables and the 2-Kronecker quiver*", in this volume.