Paper

On Diophantine Equation $a^{x_2} - a^{x_1} = b^{y_2} - b^{y_1}$

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In this paper, we study the diophantine equation $a^{x_2} - a^{x_1} = b^{y_2} - b^{y_1}$. This equation is rewritten to the diophantine equation $(a^{x_{12}} - 1)/b^{y_1} = (b^{y_{12}} - 1)/a^{x_1}$. Then, by considering the factorization into prime factor of $b^{y_{12}} - 1$, we find solutions of the equation. In the case of $b^{2m} - 1 = l_1^{s_1}$,the equation has two solutions. In the case of $b^{2m} - 1 = l_1^{s_1} l_2^{s_2}$ ($m \neq 1$), the the equation has two solutions. In the case of $b^{2m} - 1 = l_1^{s_1} l_2^{s_2} l_3^{s_3}$ ($m \neq 1, 2$), if m is even, the equation has no solutions. Futhermore, if m is odd, the equation has a unique solution under condituins which a is a prime number and x_{12} is even. In the case of $b^{4m} - 1 = l_1^{s_1} l_2^{s_2} l_3^{s_3} l_4^{s_4}$ ($m \neq 1$), the diophantine equation has a unique solution.

Keywords : Diophantine equation, Existence of solutions, Factorization into prime factors

1. Intoroduction

In this paper, we treat the diophantine equation

(1.1) $$a^{x_2} - a^{x_1} = b^{y_2} - b^{y_1}.$$

In the case of $x_2 = 2$ and $x_1 = y_1 = 1$, Mordell(1) proved theorem 1.1.

Theorem 1.1 Let a, b be integers.

The elliptic diophantine equation $a^2 - a^1 = b^3 - b^1$ has ten solutions

$$(a, b) = (0, 0), (0, \pm 1), (1, 0), (1, \pm 1), (3, 2), (-2, 2), (15, 6), (-14, 6).$$

Furthermore, Mignotte and Pethö(2) proved theorem 1.2.

Theorem 1.2 Let a, b, y_2 be positive integers.

When b is a prime power, the diophantine equation

$$a^2 - a^1 = b^{y_2} - b^1 \quad (y_2 > 2)$$

has five solutions $(a, b, y_2) = (3, 2, 3), (6, 2, 5), (91, 2, 13), (16, 3, 5), (280, 5, 7)$.

We suppose the following conditions:

1) All variables in this paper are positive integers,
2) Integers a, b are not powers, and let $a > b \geq 2$,
3) Let $x_2 > x_1$ and $y_2 > y_1$.

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Then, Bennett[3] give the following list of solutions on (1.1):

(1.2) $$3^2 - 3^1 = 2^3 - 2^1 ,$$

(1.3) $$3^3 - 3^1 = 2^5 - 2^3 ,$$

(1.4) $$3^5 - 3^1 = 2^8 - 2^4 ,$$

(1.5) $$5^3 - 5^1 = 2^7 - 2^3 ,$$

(1.6) $$13^3 - 13^1 = 3^7 - 3^1 ,$$

(1.7) $$91^2 - 91^1 = 2^{13} - 2^1 ,$$

(1.8) $$6^2 - 6^1 = 2^5 - 2^1 ,$$

(1.9) $$15^2 - 15^1 = 6^3 - 6^1 ,$$

(1.10) $$280^2 - 280^1 = 5^7 - 5^1 ,$$

(1.11) $$4930^2 - 4930^1 = 30^5 - 30^1 ,$$

(1.12) $$6^5 - 6^4 = 3^8 - 3^4$$

Let $x_{12} = x_2 - x_1,\ \ y_{12} = y_2 - y_1$.

When $\gcd(a,\ b) = 1$, (1.1) leads the diophantine equation

(1.13) $$\frac{a^{x_{12}} - 1}{b^{y_1}} = \frac{b^{y_{12}} - 1}{a^{x_1}} := k .$$

When $\gcd(a,\ b) = d^s (>1)$, where d is not power, (1.1) leads the diophantine equation

(1.14) $$\frac{a^{x_{12}} - 1}{B^{y_1}} = \frac{b^{y_{12}} - 1}{A^{x_1}} := k ,$$

where $a = d^u A,\ b = d^v B,\ \min\{u,\ v\} = s$ and $ux_1 = vy_1$. Furthermore $A,\ B$ do not include any prime factors of d. And, if b is a prime number, (1.14) leads

(1.15) $$a^{x_{12}} - 1 = \frac{b^{y_{12}} - 1}{A^{x_1}} := k ,$$

where $a = b^u A$.

Let $l_1,\ l_2,\ l_3,\ l_4$ be different prime numbers. Then we show the following theorems:

Theorem 1.3 Let $b^{y_{12}} - 1 = l_1^{s_1}$. When $y_{12} = 2m$, (1.1) has two solutions (1.2), (1.3).

Theorem 1.4 Let $b^{y_{12}} - 1 = l_1^{s_1} l_2^{s_2}$. When $y_{12} = 2m$ ($m \neq 1$), (1.1) has four solutions (1.4), (1.5), (1.8), (1.12).

Theorem 1.5 Let $b^{y_{12}} - 1 = l_1^{s_1} l_2^{s_2} l_3^{s_3}$. When $y_{12} = 4m$ ($m \neq 1$) , (1.1) has no solutions.

Theorem 1.6 Let $b^{y_{12}} - 1 = l_1^{s_1} l_2^{s_2} l_3^{s_3}$. When $y_{12} = 4m + 2$, if a is a prime number and x_{12} is even then (1.1) has a unique solution (1.6).

Theorem 1.7 Let $b^{y_{12}} - 1 = l_1^{s_1} l_2^{s_2} l_3^{s_3} l_4^{s_4}$. When $y_{12} = 4m$ ($m \neq 1$), (1.1) has a unique solution (1.7).

2. The proof of Theorem 1.3

We prove theorem 1.3 by using the following Catalan's theorem:

Catalan's theorem Let $\alpha,\ \beta,\ x,\ y$ be positive integers.

Then the equation $\alpha^x - \beta^y = 1$ has a unique solution $(\alpha,\ \beta,\ x,\ y) = (3,\ 2,\ 2,\ 3)$.

Let M_p be Merseme prime numbers with power p, so that $M_p = 2^p - 1$ ($p = 2,\ 3,\ 5,\ 7,\ 13,\ \cdots$).

Proposition 2.1 The equation $b^{y_{12}} - 1 = l_1^{s_1}$ ($y_{12} \neq 1$) has solutions $3^2 - 1 = 2^3$ and $2^p - 1 = M_p^{\ 1}$ ($p = 2,\ 3,\ 5,\ 7,\ 13,\ \cdots$).

Proof When $s_1 > 1$, from Catalan's theorem, the equation $b^{y_{12}} - 1 = l_1^{s_1}$ has a unique solution $3^2 - 1 = 2^3$. When $s_1 = 1$, $l_1 = (b-1) \times \{(b^{y_1} - 1)/(b-1)\}$ leads $b - 1 = 1$. Thus we have $b = 2$ and $l_1 = M_p$. □

Corollary 2.2 When $y_{12} = 2m$, the equation $b^{y_{12}} - 1 = l_1^{s_1}$ has two solutions $3^2 - 1 = 2^3$, $2^2 - 1 = 3^1$.

We remark $a > b$・2.

In the case of $\gcd(a,\ b) = 1$, from Corollary 2.2 and (1.13), we have

$$\frac{3^{x_{12}} - 1}{2^{y_1}} = \frac{2^2 - 1}{3^1} = 1,$$

so that $3^{x_{12}} - 1 = 2^{y_1}$. Thus this equation has two solutions $(x_{12},\ y_1) = (1,\ 1),\ (2,\ 3)$.

In the case of $\gcd(a,\ b) > 1$, from Corollary 2.2 and (1.15), we have

$$(2^u \cdot 3)^{x_{12}} - 1 = \frac{2^2 - 1}{3^1} = 1,$$

so that $(2^u \cdot 3)^{x_{12}} = 2$. This equation has no solutions.

Remark 2.3 Kobachi(4) prove that if $\gcd(a,\ b) = 1$ then the diophantine equation $\dfrac{a^{X_{12}} - 1}{b^{Y_1}} = \dfrac{b^{Y_{12}} - 1}{a^{X_1}} = 1$ has two solutions $\dfrac{3^1 - 1}{2^1} = \dfrac{2^2 - 1}{3^1} = 1$, $\dfrac{3^2 - 1}{2^3} = \dfrac{2^2 - 1}{3^1} = 1$.

3. The proof of Theorem 1.4

Lemma 3.1 The system of equations $\begin{cases} b^m - 1 = 1 \\ b^m + 1 = K \end{cases}$ has no solutions except $K = 3$.

Proof It is clear. □

Lemma 3.2 The system of equations $\begin{cases} b^m - 1 = 2 \\ b^m + 1 = K \end{cases}$ has no solutions except $K = 4$.

Proof It is clear. □

Lemma 3.3 The equations $b^m + 1 = 2^{r+1}$ ($m \neq 1$) has no solutions.

Proof It is clear from Catalan's theorem. □

Proposition 3.4 The equation $b^{2m}-1=l_1^{s_1}l_2^{s_2}$ ($m\neq 1$) has three solutions $2^4-1=3^1\cdot 5^1$, $2^6-1=7^1\cdot 3^2$, $3^4-1=5^1\cdot 2^4$.

Proof In the case of $b\equiv 0 \pmod 2$, we may assume $l_1^{s_1}<l_2^{s_2}$. And $\gcd(b^m-1,\ b^m+1)=1$ is satisfied. Then, from lemma 3.1, the equation $b^{2m}-1=l_1^{s_1}l_2^{s_2}$ ($m\neq 1$) leads the system of equations $\begin{cases} b^m-1=l_1^{s_1} \\ b^m+1=l_2^{s_2} \end{cases}$ ($m\neq 1$). From proposition 2.1, the equation $b^m-1=l_1^{s_1}$ has solutions $2^p-1=M_p^{\ 1}$ ($p=2,\ 3,\ 5,\ 7,\ 13,\ \cdots$). If $p>3$ is satisfied then $(2^p+1)/3$ includes at least one odd prime number except 3. Thus $2^p+1=l_2^{s_2}$ has no solutions. Therefor $b^{2m}-1=l_1^{s_1}l_2^{s_2}$ has two solutions $2^4-1=3^1\cdot 5^1$, $2^6-1=7^1\cdot 3^2$.

In the case of $b\equiv 1 \pmod 2$, we may assume $l_2^{s_2}=2^{r+2}$. And $\gcd(b^m-1,\ b^m+1)=2$ is satisfied. Then, from lemma 3.2 and lemma 3.3, the equation $b^{2m}-1=l_1^{s_1}l_2^{s_2}$ ($m\neq 1$) leads the system of equations $\begin{cases} b^m-1=2^{r+1} \\ b^m+1=2l_1^{s_1} \end{cases}$ ($m\neq 1$). From proposition 2.1, the equation $b^m-1=2^{r-1}$ has a unique solution $3^2-1=2^3$. Thus $b^{2m}-1=l_1^{s_1}l_2^{s_2}$ has a unique solution $3^6-1=5^1\cdot 2^4$. □

We remark $a>b\geq 2$.

In the case of $\gcd(a,\ b)=1$, from proposition 3.4 and (1.13), we have the following equations:

(3.1) $$\frac{3^{x_{12}}-1}{2^{y_1}}=\frac{2^4-1}{3^1}=5,$$

(3.2) $$\frac{5^{x_{12}}-1}{2^{y_1}}=\frac{2^4-1}{5^1}=3,$$

(3.3) $$\frac{7^{x_{12}}-1}{2^{y_1}}=\frac{2^6-1}{7^1}=9,$$

$$\frac{3^{x_{12}}-1}{2^{y_1}}=\frac{2^6-1}{3^2}=7, \qquad \frac{5^{x_{12}}-1}{3^{y_1}}=\frac{3^4-1}{5^1}=16.$$

If (3.1) is satisfied then $x_{12}=O_5(3)=4$, where notation $O_q(z)$ is multiplicative order of z module q. Thus $3^4-1=2^4\cdot 5^1$ follows. Therefor (3.1) has a unique solution $(x_{12},\ y_1)=(4,\ 4)$.

If (3.3) is satisfied then $x_{12}=O_9(3)=7$. Thus $7^3-1=2^1\cdot 3^2\cdot 19^1$ follows. There (3.3) has no solutions.
In the same way, we confirm that (3.1) and (3.2) each have a unique solution. Therefor (1.1) has two solutions (1.4), (1.5).
In the case of $\gcd(a,\ b)>1$, from proposition 3.4 and (1.15), we have the following equations:

(3.4) $$(2^u\cdot 3)^{x_1}-1=\frac{2^4-1}{3^1}=5,$$

(3.5) $$(2^u\cdot 5)^{x_1}-1=\frac{2^4-1}{5^1}=3,$$

$$(2^u\cdot 7)^{x_1}-1=\frac{2^6-1}{7^1}=9, \qquad (2^u\cdot 3)^{x_1}-1=\frac{2^6-1}{3^2}=7,$$

$$(2^u\cdot 9)^{x_1}-1=\frac{2^6-1}{9}=7, \qquad (3^u\cdot 5)^{x_1}-1=\frac{3^4-1}{5^1}=16,$$

(3.6) $$(3^u \cdot 2)^{x^1} - 1 = \frac{3^4 - 1}{2^4} = 5,$$

$$(3^u \cdot 4)^{x^1} - 1 = \frac{3^4 - 1}{4^2} = 5, \qquad (3^u \cdot 16)^{x^1} - 1 = \frac{3^4 - 1}{16} = 5$$

If (3.4) is satisfied then $(2^u \cdot 3)^{x_1} = 6$. Thus (3.4) has a unique solution $(u,\ x_1) = (1,\ 1)$.

If (3.5) is satisfied then $(2^u \cdot 5)^{x_1} = 4$. Thus (3.5) has no solutions.

In the same way, we confirm that (3.4) and (3.6) each have a unique solution. Thus (1.1) has two solutions (1.8), (1.12).

4. The proof of Theorem 1.5

Lemma 4.1 Let l be a prime number. The system of equations $\begin{cases} b^{2m} - 1 = l^s \\ b^{2m} + 1 = K \end{cases}$ ($m \neq 1$) has no solutions except $K = 5,\ 10$.

Proof From corollary 2.2, the equation $b^{2m} - 1 = l^s$ has two solutions $3^2 - 1 = 2^3$, $2^2 - 1 = 3^1$. Thus $K = 10,\ 5$ are obtained. □

Proposition 4.2 The equation $b^{4m} - 1 = l_1^{s_1} l_2^{s_2} l_3^{s_3}$ ($m \neq 1$) has two solutions $2^8 - 1 = 3^1 \cdot 5^1 \cdot 17^1$, $3^8 - 1 = 2^5 \cdot 5^1 \cdot 41^1$.

Proof In the case of $b \equiv 0 \pmod 2$, we may assume $2 < l_1^{s_1} < l_2^{s_2} < l_3^{s_3}$. And $\gcd(b^{2m} - 1,\ b^{2m} + 1) = 1$ is satisfied. From lemma 3.1 and lemma 4.1, the equation $b^{4m} - 1 = l_1^{s_1} l_2^{s_2} l_3^{s_3}$ ($m \neq 1$) leads the system of equations $\begin{cases} b^{2m} - 1 = l_1^{s_1} l_2^{s_2} \\ b^{2m} + 1 = l_3^{s_3} \end{cases}$ ($m \neq 1$).

Furthermore, from proposition 3.4, the equation $b^{2m} - 1 = l_1^{s_1} l_2^{s_2}$ ($m \neq 1$) has two solutions $2^4 - 1 = 3^1 \cdot 5^1$, $2^6 - 1 = 7^1 \cdot 3^2$. Thus the equation $b^{4m} - 1 = l_1^{s_1} l_2^{s_2} l_3^{s_3}$ ($m \neq 1$) has a unique solution $2^8 - 1 = 3^1 \cdot 5^1 \cdot 17^1$.

In the case of $b \equiv 1 \pmod 2$, we may assume $l_3^{s_3} = 2^{r+2}$ and $2 < l_1^{s_1} < l_2^{s_2}$. Furthermore $\gcd(b^{2m} - 1,\ b^{2m} + 1) = 2$ and $v_2(b^{2m} + 1) = 1$ are satisfied. From lemma 4.1, the equation $b^{4m} - 1 = l_1^{s_1} l_2^{s_2} l_3^{s_3}$ ($m \neq 1$) leads the system of equations $\begin{cases} b^{2m} - 1 = 2^{r+1} l_1^{s_1} \\ b^{2m} + 1 = 2 l_2^{s_2} \end{cases}$ ($m \neq 1$). And, from Proposition 3.4, the equation $b^{2m} - 1 = 2^{r+1} l_1^{s_1}$ ($m \neq 1$) has a unique solution $3^4 - 1 = 2^4 \cdot 5^1$.

Thus the equation $b^{4m} - 1 = l_1^{s_1} l_2^{s_2} l_3^{s_3}$ ($m \neq 1$) has a unique solution $3^8 - 1 = 2^5 \cdot 5^1 \cdot 41^1$. □

We remark $a > b \geq 2$.

In the case of $\gcd(a,\ b) = 1$, from proposition 4.2 and (1.13), we have the following equations:

$$\frac{3^{x_{12}} - 1}{2^{y_1}} = \frac{2^8 - 1}{3^1} = 85, \qquad \frac{85^{x_{12}} - 1}{2^{y_1}} = \frac{2^8 - 1}{85^1} = 3,$$

$$\frac{5^{x_{12}} - 1}{2^{y_1}} = \frac{2^8 - 1}{5^1} = 51, \qquad \frac{51^{x_{12}} - 1}{2^{y_1}} = \frac{2^8 - 1}{51^1} = 5,$$

$$\frac{17^{x_{12}} - 1}{2^{y_1}} = \frac{2^8 - 1}{17^1} = 15, \qquad \frac{15^{x_{12}} - 1}{2^{y_1}} = \frac{2^8 - 1}{15^1} = 17,$$

$$\frac{5^{x_{12}} - 1}{3^{y_1}} = \frac{3^8 - 1}{5^1} = 1312, \qquad \frac{1312^{x_{12}} - 1}{3^{y_1}} = \frac{3^8 - 1}{1312^1} = 5,$$

$$\frac{41^{x_{12}}-1}{3^{y_1}}=\frac{3^8-1}{41^1}=160\ , \qquad \frac{160^{x_{12}}-1}{3^{y_1}}=\frac{3^8-1}{160^1}=41\ ,$$

$$\frac{205^{x_{12}}-1}{3^{y_1}}=\frac{3^8-1}{205^1}=32\ .$$

In the case of $\gcd(a,\ b)>1$, from proposition 4.2 and (1.15), we have the following equations:

$$(2^u\cdot 3)^{x_{12}}-1=\frac{2^8-1}{3^1}=85\ , \qquad (2^u\cdot 85)^{x_{12}}-1=\frac{2^8-1}{3^1}=3\ ,$$

$$(2^u\cdot 5)^{x_{12}}-1=\frac{2^8-1}{5^1}=51\ , \qquad (2^u\cdot 51)^{x_{12}}-1=\frac{2^8-1}{51^1}=5\ ,$$

$$(2^u\cdot 17)^{x_{12}}-1=\frac{2^8-1}{17^1}=15\ , \qquad (2^u\cdot 15)^{x_{12}}-1=\frac{2^8-1}{15^1}=17\ ,$$

$$(3^u\cdot 5)^{x_{12}}-1=\frac{3^8-1}{5^1}=1312\ , \qquad (3^u\cdot 1312)^{x_{12}}-1=\frac{3^8-1}{1312^1}=5\ ,$$

$$(3^u\cdot 41)^{x_{12}}-1=\frac{3^8-1}{41^1}=160\ , \qquad (3^u\cdot 160)^{x_{12}}-1=\frac{3^8-1}{160^1}=41\ ,$$

$$(3^u\cdot 2)^{x_{12}}-1=\frac{3^8-1}{2^5}=205\ , \qquad (3^u\cdot 32)^{x_{12}}-1=\frac{3^8-1}{32^5}=205\ ,$$

$$(3^u\cdot 205)^{x_{12}}-1=\frac{3^8-1}{205^5}=32\ .$$

Thus (1.1) has no solutions.

5. The proof of Theorem 1.6

Lemma 5.1 If $b^{4m+2}-1=l_1^{s_1}l_2^{s_2}l_3^{s_3}$ then $b=2,\ 3$.

Proof We have

(5.1) $$b^{4m+2}-1=(b^2-1)\times\frac{b^{4m+2}-1}{b^2-1}=(b^2-1)\times\frac{b^{2m+1}-1}{b-1}\times\frac{b^{2m+1}+1}{b+1}\ .$$

From $b^2\geq 4$, there exists a prime number l with $l \bullet b^2-1$ and $l\mid(b^{4m+2}-1)/(b^2-1)$. Therefore, from $\gcd((b^{2m+1}-1)/(b-1),\ (b^{2m+1}+1)/(b+1))=1$, $b^{4m+2}-1=l_1^{s_1}l_2^{s_2}l_3^{s_3}$ leads $b^2-1=l_i^{s_i'}$ （$1\leq s_i'\leq s_i$, $i\in\{1,\ 2,\ 3\}$）. Thus, from corollary 2.2, we have $b=2,\ 3$. □

We remark that x_{12} is even in this section. Put $x_{12}=2n$.

In the case of $b=2$, $l_1^{s_1}=3^1,\ l_2^{s_2}={M_p}^1$ （$p\geq 5$） and $l_3^{s_3}=(M_p+2)/3$ are satisfied from (5.1).

In (1.15), thus we have the following equations :

(5.2) $$\frac{3^{2n}-1}{2^{y_1}}=\frac{2^{2p}-1}{3^1}=\frac{M_p(M_p+2)}{3}\ ,$$

(5.3) $$\frac{{M_p}^{2n}-1}{2^{y_1}}=\frac{2^{2p}-1}{{M_p}^1}=M_p+2\ ,$$

(5.4) $$\frac{l_3^{2n}-1}{2^{y_1}}=\frac{2^{2p}-1}{l_3^{s_3}}=3M_p .$$

When (5.2) is satisfied, we have $3^{2n}-1=2^{y_1}\cdot M_p^{\ 1}\cdot l_3^{s_3}$. From proposition 4.2, $n=2$ or n is odd. If $n=2$ then $2^{y_1}\cdot M_p^{\ 1}\cdot l_3^{s_3}=2^4\cdot 5^1$. We have a contradiction. If n is odd then $M_p=(3^n-1)/2$ from (5.1) and $M_p^{\ 1}>l_3^{s_3}$. Thus $2^{p+1}-1=3^n$ is obtained. We have a contradiction.

When (5.3) is satisfied, we have $M_p^{\ 2n}-1=2^{y_1}(M_p+2)$. Thus $2^{y_1+1}+1\equiv 0 \pmod{M_p}$, so that $2^p-1\,|\,2^{y_1+1}+1$ is satisfied. We have a contradiction.

When (5.4) is satisfied, we have $l_3^{2n}-1=2^{y_1}\cdot 3\cdot M_p$.From proposition 4.2 and lemma 5.1, $n=1,\ 2$ follows. If $n=1$ then $l_3^{\ 2}-1=2^{y_1}\cdot 3\cdot(3l_3^{s_3}-2)>2^{y_1}\cdot 3^2\cdot(l_3^{s_3}-1)$. Thus $s_3=1$ is obtained. Then $2^{y_1}\cdot 3\cdot M_p=\{(M_p+2)/3\}^2-1$, so that $2^{y_1}\cdot 3^3(2^p-1)=2^3\cdot(2^{p-2}+1)(2^{p-1}-1)$ is satisfied. Therefore we have $y_1=3$ and $27(2^p-1)=(2^{p-2}+1)(2^{p-1}-1)$. Furthermore, from $p\geq 5$, $27(2^p-1)=(2^{p-2}+1)(2^{p-1}-1)$ leads $-1\equiv 1 \pmod 4$. We have a contradiction. If $n=2$ then $3M_p=\frac{l_3^{\ 2}-1}{2^{y_1-1}}\cdot\frac{l_3^{\ 2}+1}{2}$, so that the system of equations $\frac{l_3^{\ 2}-1}{2^{y_1-1}}=3$ and $\frac{l_3^{\ 2}+1}{2}=M_p$ is obtained. Thus we have $3\cdot 2^{y_1-1}=2M_p-2=4(2^{p-1}-1)$. Therefore $y_1=3$ and $p=3$ follow. But the result is contradict to $p\geq 5$.

In the case of $b=3$, $l_1^{s_1}=2^3,\ l_2^{s_2}=(3^m-1)/2$ and $l_3^{s_3}=(3^m+1)/4$ are satisfied from (5.1).
In (1.15), thus we have the following equations :

(5.5) $$\frac{l_2^{2n}-1}{3^{y_1}}=\frac{3^{2m}-1}{l_2^{s_2}}=8l_3^{s_3},$$

(5.6) $$\frac{l_3^{2n}-1}{3^{y_1}}=\frac{3^{2m}-1}{l_3^{s_3}}=8l_2^{s_2}.$$

When (5.5) is satisfied, we have $l_2^{2n}-1=2^3\cdot 3^{y_1}\cdot l_3^{s_3}$. From proposition 4.2 and lemma 5.1, $n=1,\ 2$ follows. If $n=1$ then $l_2^{\ 2}-1=2^1\cdot 3^{y_1}\cdot(l_2^{s_2}+1)>2^1\cdot 3^{y_1}\cdot(l_2^{s_2}-1)$. Thus $s_2=1$ is obtained. Then $\left(\frac{3^m-1}{2}\right)^2-1=2^3\cdot 3^{y_1}\cdot\frac{3^m+1}{4}$ leads $3(3^{m-1}-1)=2^3\cdot 3^{y_1}$. Thus we have $y_1=1$ and $m=3$. Therefore (5.5) has a solution

(5.7) $$\frac{13^2-1}{3^1}=\frac{3^6-1}{13^1}=56.$$

If $n=2$ then $3^{y_1}\cdot l_3^{s_3}=\frac{l_2^{\ 2}-1}{4}\cdot\frac{l_2^{\ 2}+1}{2}$, so that the system of equations $\frac{l_2^{\ 2}-1}{4}=3^{y_1}$ and $\frac{l_2^{\ 2}+1}{2}=l_3^{s_3}$ is satisfied. Thus $l_3^{s_3}-2\cdot 3^{y_1}=1$, so that $3^{y_1}(3^{m-y_1}-8)=3$ is obtained. Therefore we have $y_1=1$ and $m=3$. Furthermore $l_2^{\ 2}=13$ follows. We have a contradiction.

When (5.6) is satisfied, we have $l_3^{2n}-1=2^3\cdot 3^{y_1}\cdot l_2^{s_2}$. From proposition 4.3 and lemma 5.1, $n=1,\ 2$ follows. If $n=1$ then $l_3^{\ 2}-1=2^3\cdot 3^{y_1}\cdot(2l_3^{s_2}-1)>2^4\cdot 3^{y_1}\cdot(l_3^{s_2}-1)$. Thus $s_2=1$ is obtained. Then $\left(\frac{3^m+1}{4}\right)^2-1=2^3\cdot 3^{y_1}\cdot\frac{3^m-1}{2}$ leads $3(3^{m-1}-1)(3^m+5)=2^6\cdot 3^{y_1}\cdot(3^m-1)$. Thus we have $y_1=1$ and $(3^{m-1}-1)(3^m+5)=2^6\cdot(3^m-1)$. Furthermore

$(3^{m-1}-1)(3^m+5)=2^6\cdot(3^m-1)$ leads $1\equiv -1 \pmod 3$. We have a contradiction. If $n=2$ then $3^{y_1}\cdot l_2^{s_2}=\dfrac{l_3^2-1}{4}\cdot\dfrac{l_3^2+1}{2}$, so that the system of equations $\dfrac{l_3^2-1}{4}=3^{y_1}$ and $\dfrac{l_3^2+1}{2}=l_2^{s_3}$ is satisfied. Thus $l_2^{s_2}-2\cdot 3^{y_1}=1$, so that $3^{y_1}(3^{m-y_1}-4)=5$ is obtained. We have a contradiction.

6. The proof of Theorem 1.7

Proposition 6.1 In the case of $b\equiv 0 \pmod 2$, the equation $b^{4m}-1=l_1^{s_1}l_2^{s_2}l_3^{s_3}l_4^{s_4}$ ($m\neq 1$) has two solutions $2^{12}-1=3^2\cdot 5^1\cdot 7^1\cdot 13^1$, $2^{16}-1=3^1\cdot 5^1\cdot 17^1\cdot 257^1$.

Proof From lemma 3.1 and Lemma 4.1 and $\gcd(b^{2m}-1,\ b^{2m}+1)=1$, we have

$$(6.1)\qquad \begin{cases} b^{2m}-1=l_1^{s_1}l_2^{s_2} \\ b^{2m}+1=l_3^{s_3}l_4^{s_4} \end{cases},$$

$$(6.2)\qquad \begin{cases} b^{2m}-1=l_1^{s_1}l_2^{s_2}l_3^{s_3} \\ b^{2m}+1=l_4^{s_4} \end{cases}.$$

If (6.1) is satisfied, from proposition 3.4, $b^{2m}-1=l_1^{s_1}l_2^{s_2}$ has two solutions $2^4-1=3^1\cdot 5^1$, $2^6-1=3^2\cdot 7^1$. When $2^4-1=3^1\cdot 5^1$ is satisfied, $l_3^{s_3}l_4^{s_4}=17$ follows. We have a contradiction. When $2^6-1=3^2\cdot 7^1$, $l_3^{s_3}l_4^{s_4}=65=5^1\cdot 13^1$ follows. Thus $b^{4m}-1=l_1^{s_1}l_2^{s_2}l_3^{s_3}l_4^{s_4}$ has a solution $2^{12}-1=3^2\cdot 5^1\cdot 7^1\cdot 13^1$. If (6.2) is satisfied, from proposition 4.2 and lemma 5.2, we have the follows:

i) $b^{2m}-1=l_1^{s_1}l_2^{s_2}l_3^{s_3}$ has a solution $2^8-1=3^1\cdot 5^1\cdot 17^1$. Then $l_4^{s_4}=257^1$ follows. Thus $b^{4m}-1=l_1^{s_1}l_2^{s_2}l_3^{s_3}l_4^{s_4}$ has a solution $2^{16}-1=3^1\cdot 5^1\cdot 17^1\cdot 257^1$.

ii) $b=2$ and $m\equiv 1 \pmod 2$ are satisfied. Then $l_4^{s_4}=4^m+1=5^1\times\{(4^m+1)/5\}$ follows. Since there exists an odd prime number $l\neq 5$ with $l\mid(4^m+1)/5$, this result does not occur. □

Proposition 6.2 In the case of $b\equiv 1 \pmod 2$, the equation $b^{4m}-1=l_1^{s_1}l_2^{s_2}l_3^{s_3}l_4^{s_4}$ ($m\neq 1$) has no solutions.

Proof We may assume $l_4^{s_4}=2^{r+2}$. Furthermore $\gcd(b^{2m}-1,\ b^{2m}+1)=2$ and $\nu_2(b^{2m}+1)=1$ are satisfied. From lemma 4.1, we have

$$(6.3)\qquad \begin{cases} b^{2m}-1=2^{r+1}l_1^{s_1} \\ b^{2m}+1=2l_2^{s_2}l_3^{s_3} \end{cases},$$

$$(6.4)\qquad \begin{cases} b^{2m}-1=2^{r+1}l_1^{s_1}l_2^{s_2} \\ b^{2m}+1=2l_3^{s_3} \end{cases}$$

If (6.3) is satisfied, from Proposition 3.4, $b^{2m}-1=2^{r+1}l_1^{s_1}$ has a solution $3^4-1=2^4\cdot 5^1$.

Then $l_2^{s_2}l_3^{s_3}=41$ follows. We have a contradiction. If (6.4) is satisfied, from proposition 4.2 and lemma 5.2, we have the follows:

i) $b^{2m}-1=l_1^{s_1}l_2^{s_2}l_3^{s_3}$ has a solution $3^8-1=2^5\cdot 5^1\cdot 41^1$. Then $l_4^{s_4}=3281=17^1\times 193^1$ follows. We have a contradiction.

ii) $b=3$ and $m\equiv 1 \pmod 2$ are satisfied. Then $l_4^{s_4}=(9^m+1)/2=5^1\times\{(9^m+1)/10\}$ follows. Since there exists an odd prime number $l \neq 5$ with $l \mid (9^m + 1)/10$, this result does not occur.

If $b^{4m} - 1 = l_1^{s_1} l_2^{s_2} l_3^{s_3} l_4^{s_4}$ ($m \neq 1$) is satisfied, from proposition 6.1 and proposition 6.2, then (1.1) has no solutions except the following case:

$$\frac{91^{x_{12}} - 1}{2^{y_1}} = \frac{2^{12} - 1}{91} = 45 . \tag{6.5}$$

And, if (6.5) is satisfied, we have $x_{12} = O_{45}(91) = 1$. Furthermore $2^{y_1} = \dfrac{91^1 - 1}{45} = 2^1$, so that $y_1 = 1$ follows. Therefore solution (1.7) is obtained.

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