Paper

High order weak Lefschetz properties, higher Hessians and Jordan types for standard graded Artinian Gorenstein algebras

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In this article, we give a criterion for high order weak Lefschetz properties of standard graded Artinian Gorenstein algebras, using the Jordan types of those algebras, which can be computed by Gondim's mixed Hessian matrices.

Keywords : The weak Lefschetz property, the high order weak Lefschetz properties, Jordan types of algebras, mixed Hessian matrices.

1. Introduction

The Lefschetz properties of Artinian graded algebras have been studied by many authors see Ref.(1), Ref.(2), Ref.(3), Ref.(4) and we have introduced the high order weak Lefschetz properties to study the strong Lefschetz properties for standard graded Artinian Gorenstein algebras see Ref.(5). In this paper, we give a criterion for high order weak Lefschetz properties of standard graded Artinian Gorenstein algebras by the Jordan types of those algebras see Theorem 6.5. We see that the Jordan type of a standard graded Artinian Gorenstein algebra can be computed by the rank matrix of the algebra which is defined by Gondim's mixed Hessian matrices see Definition 5.2, Definition 6.1 and Ref.(6).

Sections 2 through 5, these are preliminary sections. Here we give reviews of Macaulay's inverse generator, the Lefschetz properties, Jordan type of a nilpotent linear endomorphism, the high order weak Lefschetz property and mixed Hessian matrices. In section 6, first we give a definition of Jordan type of a standard graded Artinian Gorenstein algebra, which is essentially based on the Gondim's work see Ref.(7), and give a criterion for high order weak Lefschetz property of a standard graded Artinian Gorenstein algebra. In section 7, the final section, we discuss the [i]-weak Lefschetz property and the weak Lefschetz level using mixed Hessian matrices.

2. Macaulay's inverse generator and the weak Lefschetz property

Let K be a field of characteristic 0, $R = K[\underline{X}] = K[X_1, \cdots, X_n]$ be a polynomial ring in n variables and $K[\underline{X}]$ acts on another polynomial ring in n variables $E = E(\underline{x}) := K[\underline{x}] = K[x_1, \cdots, x_n]$ as differential operators defied by $X_i := \dfrac{\partial}{\partial x_i}$ $(i = 1, \cdots, n)$, sometimes we denote this by $E = K[\underline{x}] \underset{\text{diff.}}{\curvearrowleft} R = K[\underline{X}]$. Since K is a field of characteristic 0, $E(\underline{x})$ is isomorphic to the injective envelop of the residue field of $K[\underline{X}]$ in the category of graded $K[\underline{X}]$ modules. Hence we remark that degree $-i$ component of E as a graded $K[\underline{X}]$ module is $E_{-i} = K[x_1, \cdots, x_n]_i$ for each $i \in \mathbb{Z}$. The proof of the following theorem can be found in Ref.(4), Ref.(8) or Ref.(9).

Theorem 2.1. (Macaulay's inverse generator) *If A is a standard graded Artinian Gorenstein K algebra of embedding dimension less than or equal to n, then there exists a homogeneous polynomial $f \in E = K[x_1, \cdots, x_n]$ such that $A \simeq R/I$ where $I := 0 \underset{K[\underline{X}]}{:} f = \{\xi \in R = K[\underline{X}] \| \xi \cdot f = 0\}$ and we call f a 'Macaulay's inverse generator' for A.*

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Notation 2.2. Let f be a nonzero homogeneous polynomials in $E = K[\underline{x}]$. We use the following notations:

$$A(f) := K[\underline{X}] \Big/ \left(0 \underset{K[\underline{X}]}{:} f\right) \text{ and } A^{\vee}(f) = K[\underline{X}] \cdot f \subseteq E.$$

Remark 2.3. Let f be a nonzero homogeneous polynomials of $\deg f = d$ in $E = K[\underline{x}]$. Then the following hold:

(1) $A^{\vee}(f) \simeq \mathrm{Hom}_K\left(A(f), K\right) \simeq A(f)[d]$ as graded $A(f)$ modules, where $[i]$ denotes the usual i-shift functor for $i \in \mathbb{Z}$, i.e., $M[i]_j := M_{i+j}$ for any graded $A(f)$ module M and $j \in \mathbb{Z}$.

(2) From (1), we have $\dim_K A_i = \dim_K A_{d-i}$ for any $i \in \mathbb{Z}$.

(3) $L \cdot A^{\vee}(f) = A^{\vee}(Lf) \simeq A(Lf)[d-1]$, $L \cdot \left(A(f)[d]\right) = \left(L \cdot A(f)\right)[d-1]$ and $L \cdot A(f) \simeq A(Lf)$.

Definition 2.4. Let $A = A(f)$ be a standard graded Artinian Gorenstein K algebra. We say that A has the '*weak Lefschetz property'* (WLP) if there exists a nonzero linear form $L \in K[\underline{X}]_1$ such that the K linear map $\times L : A \to A[1]$ has the maximal rank property, i.e., $\mathrm{rank}\left(\times L : A_i \to A_{i+1}\right) = \min\left\{\dim_K A_i, \dim_K A_{i+1}\right\}$ for all $i \in \mathbb{Z}$, where $\times L$ denote the map induced from multiplication by L. In this case, we say that A has the WLP with respect to $L \in K[\underline{X}]_1$.

3. Jordan types of nilpotent linear endmorpisms

Let (V, g) be a pair of a finite dimensional K-vector space V and a nilpotent linear endomorphism g on V with $g^e \neq 0$ and $g^{e+1} = 0$. Then V has a direct sum decomposition $V \simeq \bigoplus_{i=0}^{r-1} V_i$ into cyclic g invariant subspaces, i.e., cyclic $K[g] \simeq K[t]/(t^{e+1})$ modules with $\dim_K V_0 \geq \dim_K V_1 \geq \cdots \geq \dim_K V_{r-1} \geq 1$. Here we introduce the following definitions.

Definition 3.1. The Jordan type $J_V(g)$ of (V, g) is the partition defined as follows:

$$J_V(g) := \dim_K V_0 \oplus \dim_K V_1 \oplus \cdots \oplus \dim_K V_{r-1}.$$

We also define $\mathrm{r}_V(g)$ and $\Delta_V(g)$ the rank vector and the Δ-rank vector of (V, g) as a $e+1$-dimensional integer vector with entries:

$$\mathrm{r}_V(g)_i := \mathrm{rank}\left(g^i\right) \quad (i = 0, \cdots, e) \text{ and}$$

$$\Delta_V(g)_i := \mathrm{rank}\left(g^i\right) - \mathrm{rank}\left(g^{i+1}\right) \quad (i = 0, \cdots, e),$$

where $\mathrm{rank}\left(g^i\right) := \dim_K \mathrm{Im}\left(g^i\right)$ ($i = 0, \cdots, e$), especially $\mathrm{rank}\left(g^0\right) = \mathrm{rank}\left(\mathrm{Id}_V\right) = \dim_K V$.

Notation 3.2. Let $v = (v_i)_{i=0,\cdots,e}$ be a $e+1$-dimensional vector whose entries are all nonnegative integers, we denote

$$\lambda(v) := v_{i_0} \oplus \cdots \oplus v_{i_e}$$

the partition obtained from v, where $v_{i_0} \geq \cdots \geq v_{i_e}$ and denote $\lambda^{\vee}(v) := \left(\lambda(v)\right)^{\vee}$ the dual partition of $\lambda(v)$.

Lemma 3.3. $J_V(g) = \lambda^{\vee}\left(\Delta_V(g)\right)$ *and* $\Delta_V(g)_0 \geq \cdots \geq \Delta_V(g)_e$.

Proof. It is enough to show that $J_V(g)^\vee = \lambda(\Delta_V(g))$. Let $J_V(g) = m_0 \oplus m_1 \oplus m_2 \oplus \cdots \oplus m_{r-1}$ with $e = m_0 \geq \cdots \geq m_{r-1}$, and $J_V(g)^\vee = n_0 \oplus n_1 \oplus n_2 \oplus \cdots \oplus n_{e-1}$ with $r = n_0 \geq \cdots \geq n_{e-1}$ then we remark that

$$V \simeq K[t]/(t^{m_0+1}) \oplus K[t]/(t^{m_1+1}) \oplus K[t]/(t^{m_2+1}) \oplus \cdots \oplus K[t]/(t^{m_{r-1}+1}) \text{ as } K[g] \simeq K[t]/(t^{e+1}) \text{ modules.}$$

Hence $r = n_0 = \dim_K \mathrm{Ker}(V \xrightarrow{g} V) = \dim_K V - \dim_K g(V) = \dim_K g^0(V) - \dim_K g(V) = \Delta_V(g)_0$ and $J_{g(V)}(g)^\vee = n_1 \oplus n_2 \oplus \cdots \oplus n_{e-1}$. Similarly $n_1 = \dim_K \mathrm{Ker}(g(V) \xrightarrow{g} g(V)) = \dim_K g(V) - \dim_K g^2(V) = \Delta_V(g)_1$ and $J_{g^2(V)}(g)^\vee = n_2 \oplus \cdots \oplus n_{e-1}$ and so on. □

4. The high order Lefschetz properties and the Jordan type

First we recall that the definition of the weak Lefschetz property of order $c \geq 1$ in case of the Gorenstein standard graded Artinian algebra $A = A(f)$ with homogeneous polynomial $f \in K[\underline{x}]$.

Definition 4.1. We say that $A = A(f)$ has the '*weak Lefschetz property of order* $c \geq 1$' if there exists a nonzero linear form $L \in K[\underline{X}]_1$ such that for each $i = 1, \cdots, c$, $A(L^{i-1}f)$ has the weak Lefschetz property with respect to $L \in K[\underline{X}]_1$.

Notation 4.2. Let $L \in K[\underline{X}]_1$. We denote the Jordan type of multiplication map $\times L$ on $A(f)$ by

$$J_f(L) := J_{A(f)}(\times L),$$

and denote the rank vector and the Δ-rank vector of $(A(f), \times L)$ by $\mathrm{r}_f(L) = \mathrm{r}_{A(f)}(\times L)$ and $\Delta_f(L) := \Delta_{A(f)}(\times L)$.

Remark 4.3. $(J_f(L))_i^\vee = \Delta_f(L)_i = \dim A(L^i f) - \dim A(L^{i+1} f) = \Delta_{L^i f}(L)_0 = (J_{L^i f}(L))_0^\vee$ for $i = 0, \cdots, d = \deg f$ where $L^0 f := f$.

As a corollary of Lemma 3.3, we have the following result.

Corollary 4.4. $J_f(L) = \lambda^\vee(\Delta_f(L))$.

Notation 4.5. $h_f = h_{A(f)}$ denotes the Hilbert function of $A(f)$, i.e., $h_f(i) = h_{A(f)}(i) := \dim_K A(f)_i$ for $i \in \mathbb{Z}$ and $\underline{h_f} = \underline{h_{A(f)}}$ denotes the Hilbert vector of $A(f)$, i.e., $\underline{h_f} = (h_f(0), h_f(1), \cdots)$.

Lemma 4.6. *Let* $f \in K[\underline{x}]$ *be a nonzero homogeneous polynomial of* $\deg f = d$. *Then the following conditions are equivalent:*

(1) $A(f)$ *has the weak Lefschetz property with respect to* $L \in K[\underline{X}]_1$.

(2) $\lambda(\Delta_f(L))_0 = \lambda(\underline{h_f})_0$. (3) $(J_f(L))_0^\vee = \lambda(\underline{h_f})_0$.

Proof. It is well know that $A(f)$ has the weak Lefschetz property with respect to $L \in K[\underline{X}]_1$ if and only if

$$\dim_K \operatorname{Ker}\left(A(f) \xrightarrow{\times L} A(f)\right) = \max h_f .$$

Clearly $\max h_f = \lambda\left(\underline{h_f}\right)_0$ and $\left(J_f(L)\right)_0^\vee = \lambda\left(\Delta_f(L)\right)_0 = \dim_K \operatorname{Ker}\left(A(f) \xrightarrow{\times L} A(f)\right)$ by Corollary 4.4. This complete the proof. □

Notation 4.7. For a vector $v = (v_0, v_1, \cdots)$ and an integer $0 \le p$, we denote $v_{\le p} := (v_0, v_1, \cdots, v_p)$ and $v_{\ge p} := (v_p, v_{p+1}, \cdots)$.

Lemma 4.8. *$A = A(f)$ has the weak Lefschetz property of order c if and only if*

$$\lambda\left(\underline{h_f}\right)_{\ge i} = \lambda\left(\underline{h_{Lf}}\right)_{\ge i-1} = \cdots = \lambda\left(\underline{h_{L^i f}}\right)_{\ge 0} \quad \textit{for all } 1 \le i \le c .$$

Especially in this case, we have $\lambda\left(\underline{h_f}\right)_i = \lambda\left(\underline{h_{Lf}}\right)_{i-1} = \cdots = \lambda\left(\underline{h_{L^i f}}\right)_0$ *for* $1 \le i \le c$.

Proof. $A = A(f)$ has the weak Lefschetz property if and only if $\lambda\left(\underline{h_f}\right)_{\ge 1} = \lambda\left(\underline{h_{Lf}}\right)_{\ge 0}$. Moreover if $A = A(f)$ and $A(Lf)$ have the weak Lefschetz property if and only if $\lambda\left(\underline{h_f}\right)_{\ge 1} = \lambda\left(\underline{h_{Lf}}\right)_{\ge 0}$ and $\lambda\left(\underline{h_f}\right)_{\ge 2} = \lambda\left(\underline{h_{Lf}}\right)_{\ge 1} = \lambda\left(\underline{h_{L^2 f}}\right)_{\ge 0}$. Similarly we have $\lambda\left(\underline{h_f}\right)_{\ge i} = \lambda\left(\underline{h_{Lf}}\right)_{\ge i-1} = \cdots = \lambda\left(\underline{h_{L^i f}}\right)_{\ge 0}$ for all $1 \le i \le c$. □

Proposition 4.9. *Let $f \in K[\underline{x}]$ be a nonzero homogeneous polynomial of $\deg f = d$. Then the following conditions are equivalent:*

(1) *$A = A(f)$ has the weak Lefschetz property of order $c \ge 1$ with respect to $L \in K[\underline{X}]_1$.*

(2) $\lambda\left(\Delta_f(L)\right)_{\le c-1} = \lambda\left(\underline{h_f}\right)_{\le c-1}$. (3) $\left(J_f(L)\right)_{\le c-1}^\vee = \lambda\left(\underline{h_f}\right)_{\le c-1}$.

Proof. The equivalence $(2) \Leftrightarrow (3)$ is clear by Corollary 3.11. For the proof $(1) \Leftrightarrow (2)$, using Lemma 4.6, Lemma 4.8 and Remark 4.3, $A = A(f)$ has the weak Lefschetz property of order $c \ge 1$ with respect to $L \in K[\underline{X}]_1$ if and only if $\lambda\left(\Delta_f(L)\right)_i = \lambda\left(\Delta_{L^i f}(L)\right)_0 = \lambda\left(\underline{h_{L^i f}}\right)_0 = \lambda\left(\underline{h_f}\right)_i$ for $i = 0, \cdots, c-1$. This complete the proof. □

Remark 4.10. Let $f \in K[\underline{x}]$ be a nonzero homogeneous polynomial of $\deg f = d$, then clearly $\lambda\left(\underline{h_f}\right)_{d-1} = \lambda\left(\underline{h_f}\right)_d = 1$. If $A = A(f)$ has the weak Lefschetz property of order d, then $\lambda\left(\underline{h_{L^{d-1} f}}\right)_0 = \lambda\left(\underline{h_f}\right)_{d-1} = 1$ hence $L^{d-1} f$ is nonzero linear form and automatically $\lambda\left(\underline{h_{L^{d-1} f}}\right)_1 = 1$. Moreover by Lemma 3.8, we have $\lambda\left(\underline{h_{L^d f}}\right)_0 = \lambda\left(\underline{h_{L^{d-1} f}}\right)_1 = 1$. Hence $\lambda\left(\underline{h_{L^d f}}\right)_0 = \lambda\left(\underline{h_f}\right)_d = 1$.

From the above remark, we have the following corollary, since $A(f)$ has the strong Lefschetz property with respect to $L \in K[\underline{X}]_1$ if and only if $A = A(f)$ has the weak Lefschetz property of order $d = \deg f$.

Corollary 4.11. *Let* $f \in K[\underline{x}]$ *be a nonzero homogeneous polynomial. Then the following conditions are equivalent:*

(1) $A(f)$ *has the strong Lefschetz property with respect to* $L \in K[\underline{X}]_1$.

(2) $\lambda(\Delta_f(L)) = \lambda(\underline{h_f})$. (3) $(J_f(L))^{\vee} = \lambda(\underline{h_f})$.

5. Review of the mixed Hessian matrices

Let $A = A(f)$ be a graded Gorenstein Artinian algebra with its socle degree d, i.e., $d = \deg f$. We can choose a set of monomials $\mathrm{B}_A(r) = \{\alpha_1(r), \cdots, \alpha_{h_A(r)}(r)\}$ in $L \in K[\underline{X}]_r$ such that the image of $\mathrm{B}_A(r)$ in A is a K-linear base of A_r for each $r\,(0 \le r \le d)$. Since $A[d] \simeq K[\underline{X}]f = A(f)f = A^{\vee}(f) \subseteq E$ and $\dim_K A_r = \dim_K A^{\vee}(f)_{r-d}$, we have

$$A^{\vee}(f)_{r-d} = \sum_{i=1}^{\dim_K A_r} K\alpha_i(r)f$$

and $\mathrm{B}_A(r)f := \{\alpha_1(r)f, \cdots, \alpha_{h_A(r)}(r)f\}$ is a K-linear base of $A^{\vee}(f)_{r-d}$. Moreover since A is a standard graded Gorenstein Artinian algebra, there exists the K-linear base $\mathrm{B}_A^*(d-r) = \{\alpha_1^*(d-r), \cdots, \alpha_{h_A(r)}^*(d-r)\}$ of A_{d-r} for each $r\,(0 \le r \le d)$ such that

$$\alpha_i^*(d-r)\alpha_j(r) \equiv \delta_{ij}\vartheta \mod I(f) = 0 \underset{K[\underline{X}]}{:} f \quad (0 \le i, j \le h_A(r)),$$

where $\vartheta := \dfrac{1}{\alpha_1(d)f}\alpha_1(d)$ and δ_{ij} is the Kronecker's delta. But we can choose $\alpha_1(d)$ such that $\alpha_1(d)f = 1$, in this case $\vartheta = \alpha_1(d)$.

Remark 5.1. With the notations as above, the following hold:

(1) $\mathrm{B}_A^*(r)f := \{\alpha_1^*(r)f, \cdots, \alpha_{h_A(r)}^*(r)f\}$ is also a K-linear base of $A^{\vee}(f)_{r-d}$ for each $r\,(0 \le r \le d)$ since $\mathrm{B}_A^*(r)$ is a K-linear base of A_r for each $r\,(0 \le r \le d)$.

(2) We have $\alpha_i^*(d-r)\alpha_j(r)f = \delta_{ij}$ since $\vartheta f = 1$ for each $r\,(0 \le r \le d)$ and $0 \le i, j \le h_A(r) = h_A(d-r)$.

(3) $\mathrm{B}_A^*(d-r)$ is a dual base of $\mathrm{B}_A(r)f$ and $\mathrm{B}_A(r)$ is a dual base of $\mathrm{B}_A^*(d-r)f$ from (2).

Definition 5.2. (Mixed Hessian matrix Ref.(6)) With the notations as above, we define the mixed Hessian matrix as follows:

$$\mathrm{Hess}_f^{(r,s)} := \left(\alpha_i(r)\alpha_j(s)f\right)_{1 \le i \le h_A(r),\ 1 \le j \le h_A(s)} \quad (0 \le r, s \le d).$$

Notation 5.3. $L = L(\underline{a}) := a_1X_1 + \cdots + a_nX_n \quad (\underline{a} = (a_1, \cdots, a_n) \in K^n)$

Remark 5.4. We remark that there is a following commutative diagram of K-linear maps for any linear form $L = L(\underline{a})$:

$$\begin{array}{ccc} A_r & \simeq & A^{\vee}(f)_{r-d} \\ \downarrow_{\times L^i} & \circlearrowleft & \downarrow_{\times L^i} \\ A_{r+i} & \simeq & A^{\vee}(f)_{r+i-d} \end{array},$$

where $\times L^i$ is the linear map induced form the multiplication by L^i $(i \in \mathbb{Z}_{\ge 1})$.

Lemma 5.5. *If* $f \in K[\underline{x}]$ *is a nonzero homogeneous polynomial with* $\deg f = d$ *, then*

$$L^d\left(\underline{a}\right)\cdot f = d\,!\,f\left(\underline{a}\right).$$

Proof. If we can assume that $f = \underline{x}^{\underline{e}}$ is a monomial of degree d . Then we have

$$L^d\left(\underline{a}\right)\cdot f = L^d\left(\underline{a}\right)\cdot \underline{x}^{\underline{e}} = \left(a_1X_1+\cdots+a_nX_n\right)^d\cdot \underline{x}^{\underline{e}} = \left(\cdots+\frac{d\,!}{\underline{e}!}a^{\underline{e}}X^{\underline{e}}+\cdots\right)\cdot \underline{x}^{\underline{e}} = d\,!a^{\underline{e}} = d\,!\,f\left(\underline{a}\right). \quad \square$$

Lemma 5.6. *Let* $\varphi : V = \sum_{j=1}^{m} Kv_j \to W = \sum_{i=1}^{n} Kw_i$ *be a* K *-linear map from* V *with* $\dim_K V = m$ *to* V *with* $\dim_K W = n$ *. Then representation matrix of* φ *with respect to bases* $\left\{v_j\right\}_{j=1,\cdots,m}$ *and* $\left\{w_i\right\}_{i=1,\cdots,n}$ *is* $\left(w_i^*\varphi\left(v_j\right)\right)_{1\le i\le n, 1\le j\le m}$ *where* $\left\{w_i^*\right\}_{i=1,\cdots,n}$ *is the dual base of* $\left\{w_i\right\}_{i=1,\cdots,n}$ *.*

Proof. Since $\varphi\left(v_j\right) = \sum_{i=1}^{n} w_i^*\left(\varphi\left(v_j\right)\right)w_i$ for each $1 \le j \le m$, we have

$$\varphi\left(\sum_{j=1}^{m} a_j v_j\right) = \sum_{j=1}^{m} a_j\varphi\left(v_j\right) = \sum_{j=1}^{m}\sum_{i=1}^{n} a_j w_i^*\left(\varphi\left(v_j\right)\right)w_i\,.$$

This complete the proof. $\square$

Theorem 5.7.(Gondim Ref.(6)) $i!\mathrm{Hess}_f^{(d-j-i,\,j)}\Big|_{\underline{x}=\underline{a}}$ *is the matrix associated to the map* $A^\vee\left(f\right)_{j-d} \xrightarrow{\times L(\underline{a})^i} A^\vee\left(f\right)_{j+i-d}$ *with respect to the bases* $\mathrm{B}_A\left(j\right)f$ *and* $\mathrm{B}_A^*\left(j+i\right)f$.

Proof. Since $\mathrm{B}_A\left(d-j-i\right)$ is a dual base of $\mathrm{B}_A^*\left(j+i\right)f$ and $\deg\left(\alpha_p\left(d-j-i\right)\alpha_q\left(j\right)f\right) = i$, applying the above lemma, we have the representation matrix of the map $A^\vee\left(f\right)_{j-d} \xrightarrow{\times L(\underline{a})^i} A^\vee\left(f\right)_{j+i-d}$ with respect to the bases $\mathrm{B}_A\left(j\right)f$ and $\mathrm{B}_A^*\left(j+i\right)f$:

$$\left(\alpha_p\left(d-j-i\right)\left(L\left(\underline{a}\right)^i\left(\alpha_q\left(j\right)f\right)\right)\right)_{p,q} = \left(\left(L\left(\underline{a}\right)^i\alpha_p\left(d-j-i\right)\alpha_q\left(r\right)\right)f\right)$$

$$= i!\left(\alpha_p\left(d-j-i\right)\alpha_q\left(j\right)f\right)\Big|_{\underline{x}=\underline{a}} = i!\mathrm{Hess}_f^{(d-j-i,\,j)}\Big|_{\underline{x}=\underline{a}}. \quad \square$$

Corollary 5.8. $\mathrm{rank}\left(A\left(f\right)_j \xrightarrow{\times L(\underline{a})^i} A\left(f\right)_{j+i}\right) = \mathrm{rank}\left(\mathrm{Hess}_f^{(d-j-i,\,j)}\Big|_{\underline{x}=\underline{a}}\right)$ for $0 \le i, j \le d$, where $\mathrm{Hess}_f^{(d-j-i,\,j)} := 0$ if $d < i+j$.

Proof. This follows from Theorem 5.7, since the rank of a matrix is the same as the rank of its transpose. $\square$

Notation 5.9. Let $f \in E = K[\underline{x}]$ be a nonzero homogeneous polynomial of $\deg f = d$.
We denote by $\mathrm{rank}_{K(\underline{X})}\left(\mathrm{Hess}_f^{(d-j-i,\,j)}\right)$ the rank of matrix $\mathrm{Hess}_f^{(d-j-i,\,j)}$ as a matrix with entries in the field $K(\underline{X})$ for $0 \le i, j \le d$, where $\mathrm{Hess}_f^{(d-j-i,\,j)} := 0$ if $d < i+j$.

6. The Jordan type and a criterion for the high order weak Lefschetz properties

Let $f \in E = K[\underline{x}]$ be a nonzero homogeneous polynomial of $\deg f = d$. We fix some notations and definitions as follows:

Definition 6.1. Let $r_{ij} = r_f(i,j) := \operatorname{rank}_{K(\underline{X})}\left(\operatorname{Hess}_f^{(d-j-i,\, j)}\right)$ for $0 \le i, j \le d$ and let $\mathbf{1} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{Z}^d$.

(1) The rank matrix of Gorenstein algebra $A(f)$: $\Gamma_f := \left(r_f(i,j)\right)_{0 \le i,j \le d}$.

(2) The rank vector of Gorenstein algebra $A(f)$: $\mathrm{r}_f := \Gamma_f \cdot \mathbf{1}$, i.e., $\left(\mathrm{r}_f\right)_i = \sum_{j=0}^{d} \gamma_f(i,j)$ for $0 \le i \le d$.

(3) The Δ-rank vector of of Gorenstein algebra $A(f)$: Δ_f a vector with entries $\left(\Delta_f\right)_i := \left(\mathrm{r}_f\right)_i - \left(\mathrm{r}_f\right)_{i+1}$ for $0 \le i \le d$ where $\left(\mathrm{r}_f\right)_{d+1} = 0$.

(4) The Jordan type of Gorenstein algebra $A(f)$: $J_f := \lambda^{\vee}\left(\Delta_f\right)$.

Remark 6.2. The following hold:

(1) $\left\{ L(\underline{a}) = a_1 X_1 + \cdots + a_n X_n \in K[\underline{X}]_1 \simeq \mathbb{A}_n(K) \,\middle|\, \gamma_f(i,j) = \operatorname{rank}\left(\operatorname{Hess}_f^{(d-j-i-1,\, j)}\Big|_{\underline{x}=\underline{a}}\right)\right\}$ is a nonempty Zariski open set.

(2) From (1), $\gamma_f(i,j) = \operatorname{rank}_{K(\underline{X})}\left(\operatorname{Hess}_f^{(d-j-i,\, j)}\right) = \operatorname{rank}\left(A_j(f) \xrightarrow{\times L(\underline{a})} A_{j+i}(f)\right)$

for general element $L(\underline{a}) = a_1 X_1 + \cdots + a_n X_n \in K[\underline{X}]_1$.

(3) From (2), $\mathrm{r}_f = \mathrm{r}_f\left(\times L(\underline{a})\right)$ and $\Delta_f = \Delta_f\left(\times L(\underline{a})\right)$ for general element $L(\underline{a}) = a_1 X_1 + \cdots + a_n X_n \in K[\underline{X}]_1$.

Example 6.3. Let $f = x^2u + xyv + y^2w \in K[x,y,u,v,w] \underset{\text{diff.}}{\curvearrowleft} K[X,Y,U,V,W]$. Then $\underline{h_f} = (1, 5, 5, 1)$ and a monomial base of $A = A(f)$ is depicted bellow:

$$\begin{array}{ccccccccccc}
 & & & \boxed{1} & & & & & & & \\
 & \boxed{x} & & & \boxed{y} & & \boxed{u} & & \boxed{v} & & \boxed{w} \\
\boxed{x^2} & & \boxed{xy} & & \boxed{y^2} & & \boxed{xu} & & \boxed{yw} & & \\
 & & & & & & & \boxed{xyv} & & &
\end{array}$$

Let $\{\alpha_1(0) = 1\}$, $\{\alpha_1(1) = X, \alpha_2(1) = Y, \alpha_3(1) = U, \alpha_4(1) = V, \alpha_5(2) = W\}$,

$\{\alpha_1(2) = X^2, \alpha_2(2) = XY, \alpha_3(2) = Y^2, \alpha_4(2) = XU, \alpha_5(2) = YW\}$, $\{\alpha_1(3) = XYV\}$.

Since $\operatorname{Hess}_f^{(d-i-j,\, j)} := \left(\alpha_p(d-i-j)\alpha_q(j) f\right)_{\substack{1 \le p \le \dim_K A_{d-i-j} \\ 1 \le q \le \dim_K A_j}}$, we have $\operatorname{Hess}_f^{(1,1)} = \begin{pmatrix} 2u & v & 2x & y & 0 \\ v & 2w & 0 & x & 2y \\ 2x & 0 & 0 & 0 & 0 \\ y & x & 0 & 0 & 0 \\ 0 & 2y & 0 & 0 & 0 \end{pmatrix}$,

$r_{11} = \operatorname{rank}\left(\operatorname{Hess}_f^{(3-(1+1),\, 1)}\right) = \operatorname{rank}\left(\operatorname{Hess}_f^{(1,1)}\right) = 4$, $r_{0j} = \dim_K A(f)_j = h_f(j)$ $(j = 0, \cdots, 3)$, $r_{10} = r_{12} = 1$, $r_{13} = 0$,

$r_{20} = r_{21} = 1$, $r_{22} = r_{23} = 0$, $r_{30} = 1$, $r_{31} = r_{32} = r_{33} = 0$,

$$\mathrm{r}_f=\Gamma_f\cdot\mathbf{1}=\begin{pmatrix} r_{00} & r_{01} & r_{02} & r_{03}\\ r_{10} & r_{11} & r_{12} & r_{13}\\ r_{20} & r_{21} & r_{22} & r_{23}\\ r_{30} & r_{31} & r_{32} & r_{33}\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}1&5&5&1\\1&4&1&0\\1&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}12\\6\\2\\1\end{pmatrix},\quad \Delta_f=\begin{pmatrix}6\\4\\1\\1\end{pmatrix}\ \text{and}\ J_f=4\oplus2\oplus2\oplus2\oplus1\oplus1.$$

The Ferrer's diagram of J_f is as follows:

■ ■ ■ ■ ■ ■
■ ■ ■ ■
■
■

.

Lemma 6.4. *Let* $f\in E$ *be a nonzero homogeneous polynomial of* $\deg f=d$ *and let* $m(d,1):=\left\lfloor\frac{d-1}{2}\right\rfloor$. *Then the following conditions are equivalent:*

(1) $A=A(f)$ *has the weak Lefschetz property.*

(2) $\mathrm{r}_f(1,m(d,1))=\mathrm{h}_f(m(d,1))$

(3) $\mathrm{r}_f(1,j)=\mathrm{h}_f(j)$ *for all* $j\le m(d,1)$.

(4) $\mathrm{r}_f(1,j)=\mathrm{h}_f(j)$ *for all* $j\le m(d,1)$ *and* $\mathrm{r}_f(1,j)=\mathrm{h}_f(j+1)$ *for all* $j>m(d,1)$.

(5) $\lambda(\Delta_f)_{\le0}=\lambda(\underline{h_f})_{\le0}$.

Proof. From (1) to (4), these equivalences follow from the properties of the WLP see Lemma 4.5 in Ref.(5).

$(5)\Leftrightarrow(1)$:By definition $\lambda(\Delta_f)_{\le0}=\lambda(\underline{h_f})_{\le0}$ if and only if

$$\dim_K\mathrm{Ker}(\times L)=\max(\mathrm{h}_f)=\mathrm{h}_f(m(d,1)+1)$$

for general element $L\in K[\underline{X}]_1$. This is equivalent to saying that $A=A(f)$ has the weak Lefschetz property. □

Theorem 6.5. *Let* $f\in E$ *be a nonzero homogeneous polynomial. Then the following conditions are equivalent:*

(1) $A=A(f)$ *has the weak Lefschetz property of order* c.

(2) $\lambda(\Delta_f)_{\le c-1}=\lambda(\underline{h_f})_{\le c-1}$. (3) $(J_f)^{\vee}_{\le c-1}=\lambda(\underline{h_f})_{\le c-1}$.

Proof. This follows from Proposition 4.9 and Remark 6.2. □

Definition 6.6.(see Definition 5.9 in Ref.(5)) Let f be a nonzero homogeneous polynomial of $\deg f=d$ in E. The *order of weak Lefschetz property* for f, which is denoted by $\mathrm{ord}_{\mathrm{WLP}}f$, is defined as follows:

$\mathrm{ord}_{\mathrm{WLP}}f=0$ if and only if $A=A(f)$ does not have the W.L.P.

$1\le\mathrm{ord}_{\mathrm{WLP}}f=c<d$ if and only if $A=A(f)$ has the W.L.P. of order c and does not has the W.L.P. of order $c+1$.

$\mathrm{ord}_{\mathrm{WLP}}f=\infty$ if and only if $A=A(f)$ has the S.L.P.

As a corollary of Theorem 6.5, we have the following result.

Corollary 6.7. $\mathrm{ord}_{\mathrm{WLP}} f = \max\left\{ i \in \mathbb{Z}_{\geq 0} \,\middle|\, \lambda\left(\Delta_f\right)_{\leq i-1} = \lambda\left(\underline{h_f}\right)_{\leq i-1} \right\}$, *where we assume* $\lambda\left(\Delta_f\right)_{-1} = \lambda\left(\underline{h_f}\right)_{-1} = 0$ *and* $\lambda\left(\Delta_f\right)_{\geq d+1} = \lambda\left(\underline{h_f}\right)_{\geq d+1} = 0$.

7. Criterion for the [i]-weak Lefschetz property and the weak Lefschetz level

First we recall the definition of [i]-weak Lefschetz property in Ref.(5).

Notation 7.1 Let $1 \leq d$ be a positive integer and A be a standard graded Artinian algebra.

$$m(d,i) := \left\lfloor \frac{d-i}{2} \right\rfloor \text{ if } i \in \mathbb{Z}_{\geq 0},\ m(d,i) := \infty \text{ if } i \in \mathbb{Z}_{<0} \text{ and } A_\infty := 0.$$

Definition 7.2. Let $A = A(f)$ be a standard graded Gorenstein Artinian algebra with $d = \deg f$ and let $i \in \mathbb{Z}$. We say that A has the '$[i]$*-weak Lefschetz property*' if there exists a nonzero linear form $L \in K[\underline{X}]_1$ such that the K-linear map $\times L^i : A_{m(d,i)} \to A_{m(d,i)+i}$ is injective, where $\times L^i := 0$ if $i \in \mathbb{Z}_{<0}$.

Remark 7.3. If $\times L^i : A_{m(d,i)}(f) \to A_{m(d,i)+i}(f)$ is injective then $\times L^i : A^\vee_{m(d,i)}(f) \to A^\vee_{m(d,i)+i}(f)$ is surjective.
By the symmetry of standard graded Gorenstein Artinian algebra, $\times L^i : A_{d-m(d,i)}(f) \to A_{d-(m(d,i)+i)}(f)$ is surjective. Hence $\times L^i : A_{d-m(d,i)+p}(f) \to A_{d-(m(d,i)+i)+p}(f)$ is surjective for all nonnegative integer $p \in \mathbb{Z}_{\geq 0}$. Again by the symmetry, we can see that $\times L^i : A_{m(d,i)-p}(f) \to A_{(m(d,i)+i)-p}(f)$ is injective for all nonnegative integer $p \in \mathbb{Z}_{\geq 0}$, i.e., $\times L^i : A_{\leq m(d,i)} \to A_{\leq m(d,i)+i}$ is injective.

Proposition 7.4. *Let* $f \in E$ *be a nonzero homogeneous polynomial of* $\deg f = d$. *Then the following conditions are equivalent:*
(1) $A = A(f)$ *has the* $[i]$*-weak Lefschetz property.*
(2) $\mathrm{r}_f\left(i, m(d,i)\right) = \mathrm{h}_f\left(m(d,i)\right)$. (3) $\mathrm{r}_f(i,j) = \mathrm{h}_f(j)$ *for all* $j \leq m(d,i)$.

Proof. The equivalence (1) through (3) follows from Definition 7.2 and Remark 7.3. □

Proposition 7.5. *Let* $f \in E$ *be a nonzero homogeneous polynomial of degree* $d = 2m$ *even. Then the following conditions are equivalent:*
(1) $A = A(f)$ *has the* $[2i]$*-weak Lefschetz property.*
(2) $\times L^{2i} : A(f)_{m(d,2i)} \to A(f)_{m(d,2i)+2i}$ *is injective for some* $L \in K[\underline{X}]_1$.
(3) $\det\left(\mathrm{Hess}_f^{(d-m(d,2i)-2i,\, m(d,2i))}\right) = \det\left(\mathrm{Hess}_f^{(m-i,\, m-i)}\right) \neq 0$.
In this case, automatically $\times L^{2i-1} : A(f)_{m(d,2i-1)} \to A(f)_{m(d,2i-1)+2i-1}$ *is injective, i.e.,* $A = A(f)$ *has the* $[2i-1]$*-weak Lefschetz property.*

Proof. The equivalence (1) through (3) is clear by Definition 7.2 and Corollary 5.8. □

As a corollary of Proposition 7.5, we have the following result.

Corollary 7.6. *Let $f \in E$ be a nonzero homogeneous polynomial of degree $d = 2m$. Then the following conditions are equivalent:*
(1) $A = A(f)$ *has the weak Lefschetz property of order* $2i$.

(2) $\det\left(\mathrm{Hess}_f^{(m-j,\, m-j)}\right) \neq 0$ *for all* $j \leq i$.

Proposition 7.7. *Let $f \in E$ be a nonzero homogeneous polynomial of degree $d = 2m+1$ odd.*
Then the following conditions are equivalent:
(1) $A = A(f)$ *has the* $[2i+1]$*-weak Lefschetz property.*

(2) $\times L^{2i+1} : A(f)_{m(d,2i+1)} \to A(f)_{m(d,2i+1)+2i+1}$ *is injective for some* $L \in K[\underline{X}]_1$.

(3) $\det\left(\mathrm{Hess}_f^{(d-m(d,2i+1)-(2i+1),\, m(d,2i+1))}\right) = \det\left(\mathrm{Hess}_f^{(m-i,\, m-i)}\right) \neq 0$.

In this case, automatically $\times L^{2i} : A(f)_{m(d,2i)} \to A(f)_{m(d,2i)+2i}$ *is injective, i.e.,* $A = A(f)$ *has the* $[2i]$ *-weak Lefschetz property.*

Proof. The equivalence (1) through (3) is clear by Definition 7.2 and Corollary 5.8. □

As a corollary of Proposition 7.7, we have the following result.

Corollary 7.8. *Let $f \in E$ be a nonzero homogeneous polynomial of degree $d = 2m+1$.*
Then the following conditions are equivalent:
(1) $A = A(f)$ *has the weak Lefschetz property of order* $2i+1$.

(2) $\det\left(\mathrm{Hess}_f^{(m-j,\, m-j)}\right) \neq 0$ *for all* $j \leq i$.

Definition 7.9. Let $f \in E$ be a nonzero homogeneous polynomial of degree d. We say that $A = A(f)$ has the weak Lefschetz property of *level* c if and only if $\det\left(\mathrm{Hess}_f^{(m-j,\, m-j)}\right) \neq 0$ for all $j \leq c$ where $m = m(d,0) = \left\lfloor \frac{d}{2} \right\rfloor$.

Lemma 7.10. *Let $f \in E$ be a nonzero homogeneous polynomial of degree d.*
Then the following conditions are equivalent:
(1) $A = A(f)$ *has the weak Lefschetz property of level* c.

(2) $A = A(f)$ *has the weak Lefschetz property of order* $2c+1$ *if* d *is odd*

and $A = A(f)$ *has the weak Lefschetz property of order* $2c$ *if* d *is even.*

Proof. The equivalence (1) through (2) is clear by Definition 7.2 and Corollary 5.8. □

Definition 7.11. Let $f \in E$ be a nonzero homogeneous polynomial of degree d and let $m = m(d,0) = \left\lfloor \frac{d}{2} \right\rfloor$.

The *level of weak Lefschetz property* for f, which is denoted by $l_{\mathrm{WLP}}(f)$, is defined as follows:

If $\det\left(\mathrm{Hess}_f^{(m,m)}\right) \neq 0$, then $l_{\mathrm{WLP}}(f) := \max\left\{ c \,\middle|\, \det\left(\mathrm{Hess}_f^{(m-j,m-j)}\right) \neq 0 \text{ for all } 0 \leq j \leq c-1 \right\}$.

If $\det\left(\mathrm{Hess}_f^{(m,m)}\right) = 0$, then $l_{\mathrm{WLP}}(f) := 0$.

Lemma 7.12. *Let $f \in E$ be a nonzero homogeneous polynomial of degree d and let $m = m(d,0) = \left\lfloor \frac{d}{2} \right\rfloor$. Then the following conditions are equivalent:*

(1) *$A = A(f)$ has the weak Lefschetz property of order d, i.e., $A = A(f)$ has the strong Lefschetz property.*

(2) $l_{\mathrm{WLP}}(f) = m = m(d,0)$.

Proof. The conditoin (2) eqivalently saying that $\det\left(\mathrm{Hess}_f^{(m-j,m-j)}\right) \neq 0$ for all $0 \leq j \leq m-1$. But always we have $\det\left(\mathrm{Hess}_f^{(0,0)}\right) = \det(f) \neq 0$ since $f \neq 0$. Hence (2) implies (1). The inverse implication is clear. ☐

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