Paper

High order weak Lefschetz properties for standard graded Artinian Gorenstein algebras

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In this article, we introduce the notaion of the high order weak Lefschetz properties and study those properties for trivial extensions of standard graded Artinian Gorenstein algebras.

Keywords : The strong Lefschetz property, the weak Lefschetz property, the high order weak Lefschetz property, trivial extensions.

1. Introduction

The Lefschetz properties of Artinian graded algebras have been studied by many authors see Ref.(1), Ref.(2), Ref.(3), Ref.(4).

In this article, we introduce the high order weak Lefschetz property to study the strong Lefschetz properties for graded Artinian Gorenstein algebras. The high order weak Lefschetz property of a standard graded Artinian algebra is, roughly speaking, defined by the weak Lefschetz property of its, in a sense, '*high order derived*' algebras. So, algebras have the strong Lefschetz property if they have enough high order weak Lefschetz properties. Our main results are Theorem 5.7 and Theorem 5.11.

First two sections are reviews of the Macaulay's invers system and related results. In section 2, we review the Macaulay's invers system of a standard graded Artinian algebra, which is our main tool for describing Artinian algebras. In section 3, we review the results for Macaulay's generator of a tensor product of two standard graded Artinian Gorenstein algebras. In section 4, we introduce the high order weak Lefschetz property and give some criteria. In section 5, we discuss the weak Lefschetz properties for trivial extensions of standard graded Artinian Gorenstein algebras.

2. Macaulay's invers system of a standard graded Artinian algebra

Let K be a field of characteristic 0, $K[\underline{X}]=K[X_1,\cdots,X_n]$ be a polynomial ring in n variables and $K[\underline{X}]$ acts on another polynomial ring in n variables $E=E(\underline{x}):=K[\underline{x}]=K[x_1,\cdots,x_n]$ as differential operators defied by $X_i:=\dfrac{\partial}{\partial x_i}$ $(i=1,\cdots,n)$. Since K is a field of characteristic 0, $E(\underline{x})$ is isomorphic to the injective envelop of the residue field of $K[\underline{X}]$ in the category of graded $K[\underline{X}]$ modules. Similarly let $K[\underline{U}]=K[U_1,\cdots,U_r]$ be a polynomial ring in r variables and act on another polynomial ring in r variables $E(\underline{u}):=K[\underline{u}]=K[u_1,\cdots,u_r]$ as differential operators defied by $U_i:=\dfrac{\partial}{\partial u_i}$ $(i=1,\cdots,r)$. $E(\underline{x},\underline{u}):=E(\underline{x})\underset{K}{\otimes}E(\underline{u})=K[\underline{x},\underline{u}]$ inherits the natural $K[\underline{X},\underline{U}]$ module structure.

For any standard graded K algebra of finite type $A=K[A_1]=\bigoplus_{i\in\mathbb{Z}_{\geq 0}}A_i$ with graded maximal ideal $\mathfrak{m}_A:=\bigoplus_{i>0}A_i$, we denote by $\mathrm{GrMod}(A)$ the category of graded A modules with degree preserving morphisms. Let $[i]:\mathrm{GrMod}(A)\to\mathrm{GrMod}(A)$ ($i\in\mathbb{Z}$) be the usual i-shift functor (i.e. $M[i]_j:=M_{i+j}$ for $i,j\in\mathbb{Z}$, $M\in\mathrm{GrMod}(A)$). If A is a standard graded Cohen-Macaulay K algebra, then we denote by K_A the graded canonical module of A.

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Definition 2.1. (see A3.4, Ex. A3.5, p.625 in Ref.(5)) Let M be a graded A module whose homogeneous components are all finite dimensional, we define the '*graded* K *-dual*' of M as follows:

$$M^{\vee} := \hom_{\mathrm{gr}}(M,K) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_K(M_{-i},K).$$

Notation 2.2. We denote:

$$N \underset{M}{:} I := \{\xi \in M \mid \xi I \subseteq N\} (\subseteq M)$$

for $N \subseteq M$ a inclusion of graded A modules and I a subset of A. Similarly we denote:

$$\mathrm{ann}_A(W) = 0 \underset{A}{:} W := \{\xi \in A \mid \xi W = 0\} (\subseteq A)$$

for a subset W of a graded A module M.

The following well known theorem gives us the Macaulay's invers system which is our main tool.
For the proof, see Ref.(4), Ref.(6) or Ref.(7).

Theorem 2.3. (Macaulay's invers system) *If A is a standard graded K algebra of embedding dimension less than or equal to n, then there exists a graded $K[\underline{X}]$ submodule $M \subseteq E$ such that $A \simeq R/I$ where $I := 0 \underset{K[\underline{X}]}{:} M$ and $A^{\vee} \simeq M$, especially, $K[\underline{X}]^{\vee} \simeq E$. Moreover the following hold:*

(1) *M is isomorphic to the graded injective envelop of the residue field of A and $M / \mathfrak{m}_A M \simeq \left(0 \underset{A}{:} \mathfrak{m}_A\right)^{\vee}$.*

(2) *M is a non-zero finitely generated $K[\underline{X}]$ module if and only if A is Artinian. In this case, $M \simeq K_A$.*

(3) *If $A = \bigoplus_{i=0}^{d} A_i$ is a standard graded Gorenstein Artinian algebra with $A_d \neq 0$, then $A^{\vee} \simeq M$ is a single generated graded $K[\underline{X}]$ submodule and there exists a homogeneous polynomial f in E of degree d such that $M = K[\underline{X}]\cdot f$. Especially, $A^{\vee}[-d] \simeq A$.*

Definition 2.4. If $A = \bigoplus_{i=0}^{d} A_i d$ is a standard graded Gorenstein Artinian algebra with $A_d \neq 0$, then we say that '*socle degree*' of A is d.

Notation 2.5. Let $\underline{f} = \{f_j\}_{j\in J}$ be a set of nonzero homogeneous polynomials in E.

We denote: $A^{\vee}(\underline{f}) = A^{\vee}_{K[\underline{X}]}(\underline{f}) := \sum_{j\in J} K[\underline{X}] f_j \subseteq E$;

$$I(\underline{f}) = I_{K[\underline{X}]}(\underline{f}) := 0 \underset{K[\underline{X}]}{:} A^{\vee}(\underline{f}) = 0 \underset{K[\underline{X}]}{:} \underline{f} = \bigcap_{j\in J} \mathrm{ann}_{K[\underline{X}]}(f_j);$$

$$A(\underline{f}) = A_{K[\underline{X}]}(\underline{f}) := K[\underline{X}]/I(\underline{f}) = K[\underline{X}] \Big/ \bigcap_{j\in J} \mathrm{ann}_{K[\underline{X}]}(f_j).$$

Remark 2.6. We remark the following:

(1) Let A be a standard graded K algebra. If $A \simeq A(\underline{f})$ with $\underline{f} \subseteq E$ a set of nonzero homogeneous polynomials in E, then we call $\underline{f}$ a '*Macaulay's invers system*' of A.

(2) Let $\underline{f}$ be a finite set of nonzero homogeneous polynomials in E, then $A(\underline{f})$ is an Artinian algebra with $\operatorname{emb.dim} A \le \operatorname{emb.dim} K[\underline{X}] = n$ and $A(\underline{f})^{\vee} \simeq A^{\vee}(\underline{f})$.

(3) Let f be a nonzero homogeneous polynomials of degree $\deg f = d$ in E, then $A(f)\cdot f \simeq A(f)[d]$. This means that $E = \bigoplus_{i\ge 0} E_{-i}$ as a graded $K[\underline{X}]$ module, where $E_{-i} = \{ f \in E \mid \deg f = i \}$ the vector space of degree i polynomials in E for each integer $i \ge 0$.

(4) If $A^{\vee}(f) = K[\underline{X}]\cdot f \supseteq A^{\vee}(f') = K[\underline{X}]\cdot f'$ where f and f' are nonzero homogeneous polynomials in E, then $I(f) \subseteq I(f')$ so we have

$$A^{\vee}(f) = K[\underline{X}]\cdot f = A(f)\cdot f \quad \text{and} \quad A^{\vee}(f') = K[\underline{X}]\cdot f' = A(f)\cdot f' = A(f')\cdot f'.$$

(5) $A^{\vee}_{K[\underline{U}]}(\underline{g}),\ A^{\vee}_{K[\underline{X},\underline{U}]}(\underline{h}),\ I_{K[\underline{U}]}(\underline{g}),\ I_{K[\underline{X},\underline{U}]}(\underline{h}),\ A_{K[\underline{U}]}(\underline{g})$ and $A_{K[\underline{X},\underline{U}]}(\underline{h})$ are similarly defined for two sets of nonzero homogeneous polynomials $\underline{g} \subseteq E(\underline{u})$ and $\underline{h} \subseteq E(\underline{x},\underline{u})$.

Notation 2.7. Let A be a standard graded K algebra. Given a nonzero homogeneous element $\alpha \in A$ and a graded A module M, we denote the α multiplication map on M by $\times\alpha = \times\alpha|_M : M \to M[\deg\alpha]$ where $\times\alpha(\xi) := \alpha\cdot\xi$ for $\xi \in M$. Also we denote by $\left(\times\alpha|_M\right)_i : M_i \to M_{i+\deg\alpha}$ ($i \in \mathbb{Z}$) the i-th graded component of $\times\alpha|_M$.

Notation 2.8. Let $\underline{f} = \{f_1,\cdots,f_r\}$ be a set of nonzero homogeneous polynomials in E and $L \in K[\underline{X}]_1$. We denote

$$\underline{Lf} := \{Lf_1,\cdots,Lf_r\}.$$

Lemma 2.9. *Let $\underline{f} = \{f_1,\cdots,f_r\}$ be a set of nonzero homogeneous polynomials in E and L be a nonzero linear form in $K[\underline{X}]$. Then the following hold:*

(1) $L\cdot A^{\vee}(\underline{f}) = A^{\vee}(\underline{Lf})$.

(2) $A(\underline{Lf}) \simeq A(\underline{f}) \Big/ \left(0 \underset{A(\underline{f})}{:} L\right)$.

(3) $\dim_K \operatorname{Ker}\left(\times L|_{A(\underline{f})}\right) = \dim_K A(\underline{f}) - \dim A(\underline{Lf})$.

Proof. (1): From the definition of $A^{\vee}(\underline{f})$, we have $L\cdot A^{\vee}(\underline{f}) = L\cdot\sum_{i=1}^{r} K[\underline{X}] f_i = \sum_{i=1}^{r} K[\underline{X}] Lf_i = A^{\vee}(\underline{Lf})$.

(2) : Since $I(\underline{Lf}) = 0 \underset{K[\underline{X}]}{:} L\cdot\underline{f} = \left(0 \underset{K[\underline{X}]}{:} \underline{f}\right) \underset{K[\underline{X}]}{:} L = I(\underline{f}) \underset{K[\underline{X}]}{:} L$ and $A(\underline{f}) = K[\underline{X}]/I(\underline{f})$, we get (2).

(3): Since $\operatorname{Ker}\left(\times L|_{A(\underline{f})}\right) = 0 \underset{A(\underline{f})}{:} L$, we get (3) form the following exact sequence:

$$0 \to 0 \underset{A(\underline{f})}{:} L \to A(\underline{f}) \to A(\underline{f}) \Big/ \left(0 \underset{A(\underline{f})}{:} L\right) \simeq A(\underline{Lf}) \to 0. \ \square$$

3. Tensor product of two standard graded Artinian Gorenstein algebras and its Macaulay's generator

In this section, we review the result for Macaulay's generator of a tensor product of two standard graded Artinian Gorenstein algebras.

Lemma 3.1. *Let* $f_1,\cdots,f_r \in K[\underline{x}]$ *and* $g_1,\cdots,g_s \in K[\underline{u}]$ *be linearly independent elements over* K *. Then* $\{f_i g_j\}_{1\le i\le r,\,1\le j\le s}$ *is a set of linearly independent* rs *elements in* $K[\underline{x},\underline{u}]$.

Proof. It is enough to show that $\dim_K W = rs$, where $W := \sum_{1\le i\le r,\,1\le j\le s} K f_i g_j$.

Using the natural isomorphism $\Phi : \begin{array}{ccc} K[\underline{x}]\underset{K}{\otimes}K[\underline{u}] & \to & K[\underline{x},\underline{u}] \\ f\otimes g & \mapsto & f\cdot g \end{array}$, we have

$$rs = \dim_K\left(V_1\otimes V_2\right) = \dim_K W$$

where $V_1 := \sum_{1\le i\le r,} K f_i$, $V_2 := \sum_{1\le j\le s} K g_j$ and $W = \Phi\left(V_1\otimes V_2\right)$. □

Since $K[\underline{X},\underline{U}]I_{K[\underline{X}]}(f)+K[\underline{X},\underline{U}]I_{K[\underline{U}]}(g)\subseteq I_{K[\underline{X},\underline{U}]}(f\cdot g)$, we have the following commutative diagram:

$$\begin{array}{ccc} \pi\ :\ A_{K[\underline{X}]}(f)\underset{K}{\otimes}A_{K[\underline{U}]}(g) & \twoheadrightarrow & A_{K[\underline{X},\underline{U}]}(f\cdot g) \\ \| & \circlearrowleft & \| \\ K[\underline{X},\underline{U}]\Big/K[\underline{X},\underline{U}]I_{K[\underline{X}]}(f)+K[\underline{X},\underline{U}]I_{K[\underline{U}]}(g) & \twoheadrightarrow & K[\underline{X},\underline{U}]\Big/I_{K[\underline{X},\underline{U}]}(f\cdot g) \end{array}$$

for nonzero homogeneous polynomials $f\in E(\underline{x})$ and $g\in E(\underline{u})$.

The following Lemma is well known but we give a proof for the convenience of the reader.

Lemma 3.2. $\pi\ :\ A_{K[X]}(f)\underset{K}{\otimes}A_{K[U]}(g)\to A_{K[X,U]}(f\cdot g)$ *is an isomorphism.*

Proof. Since π is surjective, it is enough to show that

$$\dim_K A_{K[X,U]}(f\cdot g) = \dim_K A_{K[X]}(f)\cdot\dim_K A_{K[U]}(g)\ .$$

We can express $A^\vee_{K[X]}(f)$ and $A^\vee_{K[U]}(g)$ as follows:

$$A^\vee_{K[X]}(f) = \sum_{i=1}^{r} K\alpha_i f \subseteq K[x]\ \text{ with }\ \alpha_i\in K[X]\left(i=0,\cdots,r:=\dim_K A_{K[X]}(f)\right);$$

$$A^\vee_{K[U]}(g) = \sum_{j=0}^{s} K\beta_j g \subseteq K[u]\ \text{ with }\ \beta_j\in K[U]\left(j=0,\cdots,s:=\dim_K A_{K[Y]}(g)\right).$$

Then we get $A^\vee_{K[X,U]}(f\cdot g) = K[X,U](f\cdot g) = \sum_{\substack{0\le i\le \deg f\\ 0\le j\le \deg g}} K\left(\alpha_i f\right)\cdot\left(\beta_j g\right)\in K[x,u]$.

Since $\alpha_1 f,\cdots,\alpha_r f\in K[x]$ and $\beta_1 g,\cdots,\beta_s g\in K[\underline{u}]$ are linearly independent elements over K , we have $\dim_K A_{K[X,U]}(f\cdot g) = \dim_K A_{K[X]}(f)\cdot\dim_K A_{K[U]}(g)$ by lemma 3.1. □

Definition 3.3. Let $M \in \mathrm{GrMod}(A)$ be an graded A module whose homogeneous components are all finite dimensional.

(1) We define the Hilbert function $h_M : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ of M by $h_M(i) = \dim_K M_i$ for $i \in \mathbb{Z}$.

(2) $\mathrm{H}_M(t) := \sum_{i \in \mathbb{Z}} h_M(i) t^i \in \mathbb{Z}[\![t]\!]$ is called the Hilbert series of M.

Moreover if we assume $M \simeq \bigoplus_{i=0}^{d} M_i$, then

(3) The maximal value of Hilbert function $\mathrm{Max}(h_M)$ is called the Spernar number of M.

(4) We call h_M is unimodal if $h_M(0) \leq \cdots \leq h_M(p) \geq \cdots \geq h_M(d)$ for some $0 \leq p \leq d$.

Notation 3.4. Let $\underline{f} \subseteq E(\underline{x})$, $\underline{g} \subseteq E(\underline{u})$ and $\underline{h} \subseteq E(\underline{x}, \underline{u})$ be subsets of nonzero homogeneous polynomials. We denote:

$$h_{\underline{f}} = h_{\underline{f}}^{K[\underline{X}]} := h_{A(\underline{f})},\quad h_{\underline{g}} = h_{\underline{g}}^{K[\underline{U}]} := h_{A(\underline{g})} \text{ and } h_{\underline{h}} = h_{\underline{h}}^{K[\underline{X},\underline{U}]} := h_{A(\underline{h})};$$

$$\mathrm{H}_{\underline{f}} = \mathrm{H}_{\underline{f}}^{K[\underline{X}]} := \mathrm{H}_{A(\underline{f})},\quad \mathrm{H}_{\underline{g}} = \mathrm{H}_{\underline{g}}^{K[\underline{U}]} := \mathrm{H}_{A(\underline{g})} \text{ and } \mathrm{H}_{\underline{h}} = \mathrm{H}_{\underline{h}}^{K[\underline{X},\underline{U}]} := \mathrm{H}_{A(\underline{h})}.$$

As a corollary of Lemma 3.2, we have the following results.

Corollary 3.5. *Let $f \in E(\underline{x})$ and $g \in E(\underline{u})$ be nonzero homogeneous polynomials. Then the following hold:*

$$\mathrm{H}_{f \cdot g}^{K[X,U]}(t) = \mathrm{H}_f^{K[X]}(t) \cdot \mathrm{H}_g^{K[U]}(t).$$

Corollary 3.6. $h_{f \cdot u}^{K[\underline{X},U]}(i) = h_f^{K[\underline{X}]}(i) + h_f^{K[\underline{X}]}(i-1)$ *for all* $i \in \mathbb{Z}$.

Proof. Since $A_{K[U]}(u) \simeq K[U]/(U^2)$, we have $\mathrm{H}_u^{K[U]}(t) = 1 + t$. Hence by Corollary. 3.5, we have $\mathrm{H}_{f \cdot u}^{K[\underline{X},U]}(t) = \mathrm{H}_f^{K[\underline{X}]}(t) \cdot \mathrm{H}_y^{K[U]}(t) = \mathrm{H}_f^{K[\underline{X}]}(t)(1+t) = \mathrm{H}_f^{K[\underline{X}]}(t) + t \cdot \mathrm{H}_f^{K[\underline{X}]}(t)$. This implies the assertion. □

Notation 3.7. Let $h_1, h_2 : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function. We define $h_1 * h_2 : \mathbb{Z} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ the convolution of h_1 and h_2 as follows: $h_1 * h_2(m) := \sum_{\substack{i+j=m \\ i,j \in \mathbb{Z}}} h_1(i) h_2(j)$ for $m \in \mathbb{Z}$.

Corollary 3.8. *Let $f \in E(\underline{x})$ and $g \in E(\underline{y})$ be nonzero homogeneous polynomials. Then the following hold:*

$$h_{f \cdot g}^{K[X,Y]} = h_f^{K[X]} * h_g^{K[Y]}.$$

4. The high order weak Lefschetz property for standard graded Artinian algebras

First we recall the definition of the weak Lefschetz property (WLP) and the strong Lefschetz property (SLP) of a standard graded Artinian K algebra.

Definition 4.1. Let A be a standard graded K algebra. Given a nonzero homogeneous element $\alpha \in A$ and a graded A module M whose components are all finite dimensional, we say that $\times\alpha|_M : M \to M[\deg\alpha]$ has the '*maximal rank*' property if each component $(\times\alpha|_M)_i : M_i \to M_{i+\deg\alpha}$ ($i \in \mathbb{Z}$) has maximal rank, i.e. $\dim_K(\times\alpha|_M(M_i)) = \max\{\dim_K M_i, \dim_K M_{i+\deg\alpha}\}$ for all $i \in \mathbb{Z}$.

Definition 4.2. Let A be a standard graded Artinian K algebra. We say that A has the '*weak Lefschetz property'* (WLP) if there exists a nonzero linear form $z \in A_1$ such that $\times z|_A : A \to A[1]$ has the maximal rank property. In this case, we say that A has WLP with respect to z. Moreover if A is a homomorphic image of a polynomial ring $K[\underline{X}]$, i.e., $A \simeq K[\underline{X}]/I$, then A is a graded $K[\underline{X}]$-module. If $L \in K[\underline{X}]_1$ is a preimage of $z \in A_1$, then we also say that A has WLP with respect to $L \in K[\underline{X}]_1$ .Similarly we say that A has the '*strong Lefschetz property*' (SLP) if there exists a nonzero linear form $z \in A_1$ such that $\times z^i|_A : A \to A[i]$ has the maximal rank property for all positive integer $i \in \mathbb{Z}_{\geq 1}$. In this case, we say that A has SLP with respect to z or $L \in K[\underline{X}]_1$, if $A \simeq K[\underline{X}]/I$ and $L \in K[\underline{X}]_1$ is a preimage of $z \in A_1$.

Any standard graded Gorenstein Artinian algebra with its socle degree d has the property that $A^\vee[-d] \simeq A$, see Theorem 2.3 (3). This symmetry leads the remarkable properties for the rank of each graded component of the K-linear map $\times z|_A$ ($z \in A_1$).

Lemma. 4.3. *Let A be a standard graded Gorenstein Artinian algebra with its socle degree d and $0 \neq z \in A_1$. Then the following hold:*

(1) *Let i, j be two integers with $i + j = d$. Then* $\operatorname{rank}\left(\times z|_A\right)_i = \operatorname{rank}\left(\times z|_A\right)_j$.

Especially $\left(\times z|_A\right)_i$ is injective (resp. surjective) if and only if $\left(\times z|_A\right)_j$ is surjective (resp. injective).

(2) *If $\left(\times z|_A\right)_j$ is surjective for some integer j, then $\left(\times z|_A\right)_i$ is surjective for all $i \geq j$.*

(3) *If $\left(\times z|_A\right)_j$ is injective for some integer j, then $\left(\times z|_A\right)_i$ is injective for all $i \leq j$.*

Proof. (1) Since $A^\vee[-d] \simeq A$, we have the following commutative diagrams:

$$\begin{array}{ccc} A & \overset{\times z|_A}{\to} & A[1] \\ \| & & \| \\ A^\vee[-d] & \overset{\times z|_{A^\vee}}{\to} & A^\vee[-d+1] \end{array} \quad \text{and} \quad \begin{array}{ccc} A_i & \overset{\left(\times z|_A\right)_i}{\to} & A_{i+1} \\ \| & \circlearrowleft & \| \\ \left(A^\vee\right)_{i-d} & \overset{\left(\times z|_{A^\vee}\right)_{i-d}}{\to} & \left(A^\vee\right)_{i-d+1} \end{array} \quad \text{for } i \in \mathbb{Z}.$$

Here we remark that $\left(\times z|_{A^\vee}\right)_{i-d} = \left(\left(\times z|_A\right)_{d-i}\right)^\vee$. From the above diagrams we have

$$\operatorname{rank}\left(\times z|_A\right)_i = \operatorname{rank}\left(\times z|_{A^\vee}\right)_{i-d} = \operatorname{rank}\left(\left(\times z|_A\right)_{d-i}\right)^\vee = \operatorname{rank}\left(\times z|_A\right)_{d-i}.$$

(2) Since $\left(\times z|_A\right)_j$ is surjective, we have $z \cdot A_j = A_{j+1} = \left(\mathfrak{m}_A\right)_1 A_j$. If $i \geq j$, then

$$z \cdot A_i = z \cdot \left(\mathfrak{m}_A\right)_{i-j} A_j = \left(\mathfrak{m}_A\right)_{i-j} \cdot \left(z \cdot A_j\right) = \left(\mathfrak{m}_A\right)_{i-j} \cdot \left(\mathfrak{m}_A\right)_1 A_j = A_{i+1}.$$

This implies $\left(\times z|_A\right)_i$ is surjective.

(3) If $\left(\times z|_A\right)_j$ is injective, then by (1), $\left(\times z|_A\right)_{d-j}$ is surjective. Hence by (2), $\left(\times z|_A\right)_{d-i}$ is surjective for all $i \leq j$.

Again applying (1), we have $\left(\times z|_A\right)_i$ is injective for all $i \leq j$. □

We gather the criteria for WLP of standard graded Artinian Gorenstein algebras in the following lemma.

Notation 4.4. Given a rational number q, we denote by $\lfloor q \rfloor$ the round down of q, i.e. $\lfloor q \rfloor := \max\{i \in \mathbb{Z} | i \le q\}$.

Lemma 4.5. *Let f be a nonzero homogeneous polynomial of $\deg f = d$ in E and $m := \left\lfloor \frac{d-1}{2} \right\rfloor$. Then the following conditions are equivalent:*

(1) $A(f)$ *has the WLP with respect to* $L \in K[\underline{X}]_1$;

(2) $\dim_K A(f) - \dim_K A(Lf) = \max h_{A(f)}$;

(3) *The* K *linear map* $\times L : A(f)_i \to A(f)_{i+1}$ *is injective for* $i \le m$.

(4) *The* K *linear map* $\times L : A(f)_i \to A(f)_{i+1}$ *is surjective for* $i \ge m+1$.

(5) *The* K *linear map* $\times L : A(f)_m \to A(f)_{m+1}$ *is injective.*

(6) $h_{A(f)}(m) = h_{A(Lf)}(m)$.

(7) $h_{A(f)}(i) = h_{A(Lf)}(i)$ *for* $i \le m$ *and* $h_{A(f)}(i+1) = h_{A(Lf)}(i)$ *for* $i \ge m+1$.

(8) $h_{A(f)}(i) = h_{A(Lf)}(i)$ *for* $i \le m$.

(9) $A(f)_{\le i} = A(Lf)_{\le i}$ *for* $i \le m$.

(10) $A^{\vee}(f)_{\ge -i} = A^{\vee}(Lf)_{\ge -i}$ *for* $i \le m$.

Proof. It is easy to see that $\dim_K \mathrm{Ker}(\times L|_A) = \dim_K A(f) - \dim_K A(Lf) \ge \max h_{A(f)}$ and the equality holds if and only if A has the unimodal Hilbert function and $\times L|_A$ has the maximal rank property. But the maximal rank property of $\times \alpha|_A$ forces the unimodal property of the Hilbert function of A. This implies that (1) is equivalent to (2). The other equivalences are easily checked by using Lemma 4.3 and Lemma 2.9 (3). □

Definition 4.6. Let A be a standard graded Artinian algebra. We say that A has the '*weak Lefschetz property of order* $c \ge 1$' if there exists a nonzero linear form $z \in A_1$ such that for each $i = 1, \cdots, c$, $A / \left(0 \underset{A}{:} z^{i-1}\right)$ ($i = 1, \cdots, c$) has the weak Lefschetz property with respect to $z \in A_1$. In this case, we call $z \in A_1$ a '*weak Lefschetz element of order* c'.

Notation 4.7. Let $1 \le d$ be a positive integer and A be a standard graded Artinian algebra.

$$m(d,i) := \left\lfloor \frac{d-i}{2} \right\rfloor \text{ if } i \in \mathbb{Z}_{\ge 0}, \quad m(d,i) := \infty \text{ if } i \in \mathbb{Z}_{<0} \text{ and } A_\infty := 0.$$

Definition 4.8. Let A be a standard graded Gorenstein Artinian algebra with its socle degree d and let $i \in \mathbb{Z}$. We say that A has the '$[i]$-*weak Lefschetz property*' if there exists a nonzero linear form $z \in A_1$ such that the K linear map $\times z^i : A_{m(d,i)} \to A_{m(d,i)+i}$ is injective, where $\times z^i := 0$ if $i \in \mathbb{Z}_{<0}$. Moreover if A has the $[i]$-weak Lefschetz property for each $i \in \Lambda$, where Λ be a subset of $\mathbb{Z}$, then we say that A has the 'Λ-*weak Lefschetz property*'.

Remark 4.9. If $i \ge d+1$, then $\times z^i : A_{m(d,i)} = 0 \to A_{m(d,i)+i}$ injective since $m(d,i) < 0$ and A has automatically the $[i]$-weak Lefschetz property. If $i < 0$, then we have $\times z^i : A_{m(d,i)} = A_\infty = 0 \to A_{m(d,i)+i} = A_\infty = 0$ and also A has automatically the $[i]$ -weak Lefschetz property. If $i=0$, then we have $\times z^0 = 1 : A_{m(d,0)} \to A_{m(d,0)}$ the identity map. Hence A has automatically the $\mathbb{Z}_{\le 0} \cup \mathbb{Z}_{\ge d+1}$ -weak Lefschetz property.

Lemma 4.10. *Let* f *be a nonzero homogeneous polynomial of* $\deg f = d$ *in* E. *Then the following conditions are equivalent:*

(1) $A(f)$ *has the WLP of order* $c \ge 1$ *with respect to* $L \in K[\underline{X}]_1$.

(2) $\times L : A(L^{i-1}f)_{m(d,i)} \to A(L^{i-1}f)_{m(d,i)+1}$ *is injective for each* $i = 1, \cdots, c$.

(3) $\times L : A(L^{i-1}f)_{\le m(d,i)} \to A(L^{i-1}f)_{\le m(d,i)+1}$ *is injective for each* $i = 1, \cdots, c$.

(4) $h_{A(L^{i-1}f)}\Big|_{\le m(d,i)} = h_{A(L^i f)}\Big|_{\le m(d,i)}$ *for each* $i = 1, \cdots, c$.

(5) $h_{A(f)}\Big|_{\le m(d,i)} = h_{A(L^i f)}\Big|_{\le m(d,i)}$ *for each* $i = 1, \cdots, c$.

(6) $h_{A(f)}(m(d,i)) = h_{A(L^i f)}(m(d,i))$ *for each* $i = 1, \cdots, c$.

(7) $\times L^i : A(f)_{\le m(d,i)} \to A(f)_{\le m(d,i)+i}$ *is injective for each* $i = 1, \cdots, c$.

(8) $\times L^i : A(f)_{m(d,i)} \to A(f)_{m(d,i)+i}$ *is injective for each* $i = 1, \cdots, c$.

Proof. We remark that $m(d,1) \ge m(d,2) \ge \cdots \ge m(d,c)$ and $h_{A(f)}(j) \ge h_{A(Lf)}(j) \ge \cdots \ge h_{A(L^{i-1}f)}(j) \ge h_{A(L^i f)}(j)$ for $j \in \mathbb{Z}$. These equivalences directly follow from Lemma 4.5. □

Remark 4.11. If $d = 2m$, then $\dim_K A(f)_{m-j} = \dim_K A(f)_{m+j}$ for $j = 0, \cdots, m$ since $A(f)$ is a standard graded Gorenstein algebra. Especially $\dim_K A(f)_{m(d,2j)} = \dim_K A(f)_{m(d,2j)+2j}$ and $m(d,2j) = m(d,2j-1)$ for $j = 1, \cdots, m$.

Similarly if $d = 2m+1$, then $\dim_K A(f)_{m-j} = \dim_K A(f)_{m+j+1}$ for $j = 0, \cdots, m$.

Especially $\dim_K A(f)_{m(d,2j+1)} = \dim_K A(f)_{m(d,2j+1)+2j+1}$ and $m(d,2j+1) = m(d,2j)$ for $j = 0, \cdots, m$.

Lemma 4.12. *Let* f *be a nonzero homogeneous polynomial of* $\deg f = d$ *in . Then the following conditions are equivalent:*

(1) $A(f)$ *has the S.L.P.*

(2) $A(f)$ *has the W.L.P. of order* d.

(3) $A(f)$ *has the W.L.P. of order* i *for each* $i \in \mathbb{Z}_{\ge 1}$

Proof. $(1) \Rightarrow (2)$: If $d = 2m$, then $\times L^{2j} : A(f)_{m(d,2j-1)} = A(f)_{m(d,2j)} \to A(f)_{m(d,2j)+2j}$ is an isomorophism for each $j = 1, \cdots, m$. Hence $\times L^{2j-1} : A(f)_{m(d,2j-1)} \to A(f)_{m(d,2j-1)+2j-1}$ is injective for each $j = 1, \cdots, m$. It can be shown in a similar way when $d = 2m+1$. $(2) \Rightarrow (1)$: If $d = 2m$, then $\times L^{2j} : A(f)_{m(d,2j-1)} = A(f)_{m(d,2j)} \to A(f)_{m(d,2j)+2j}$ is injective, hence an isomorphism for each $j = 1, \cdots, m$. This implies $A(f)$ has the S.L.P. in the narrow sense. $(3) \Rightarrow (2)$:This is clear. $(2) \Rightarrow (3)$: This follows by Remark 4.9. This complete the proof. □

5. The high order WLP's for trivial extensions of standard graded Artinian Gorenstein algebras

Our main results are Theorem 5.7 and Theorem 5.11. Before going into that, we need some preparations.

Notation 5.1. Let j,r be non-negative integers. We denote the binomial coefficient by ${}_rC_j := \dfrac{r!}{(r-j)!\cdot j!}$ for $0 \le j \le r$ and ${}_rC_j := 0$ if $j > r$.

Definition 5.2. Let i,e be positive integers. We define $M^i_{e,p}$ the square matrix of size $e+1$ with entries binomial coefficients as follows:

$$M^i_{e,p} := \begin{pmatrix} {}_iC_p & {}_iC_{p-1} & \cdots & {}_iC_{2p-e} \\ {}_iC_{p+1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & {}_iC_{p-1} \\ {}_iC_e & \cdots & {}_iC_{p+1} & {}_iC_p \end{pmatrix} \left(0 \le p \le e, i \ge p\right).$$

We need the following fact that $M^i_{e,p}$ is a non-singular matrix. A proof of this result can be found in [8].

Lemma 5.3. (see Lemma 6.6 in Ref.(8)) *Let* i,e *be positive integers.* $\det\left(M^i_{e,p}\right) \ne 0$ *for* $0 \le p \le e,\, i \ge p$.

Lemma 5.4. *Let* f *be a nonzero homogeneous polynomial of* $\deg f = d$ *in* E, $A := A_{K[\underline{X}]}(f)$, $B := A_{K[\underline{X},U]}\left(f \cdot u^e\right)$, $L \in K[\underline{X}]_1$ *and* $e,i \ge 1$ *positive integers. Then the following hold:*

(1) $B \simeq A \underset{K}{\otimes} K[U]\big/\left(U^{e+1}\right) = A \oplus A\overline{U} \oplus \cdots \oplus A\overline{U^e}$.

(2) *There exists the following commutative diagram:*

$$\begin{array}{ccc} B & \overset{\times(L+U)}{\rightarrow} & B[1] \\ \| & \circlearrowleft & \| \\ A \oplus A[-1] \oplus \cdots \oplus A[-e] & \overset{\Phi}{\rightarrow} & A[1] \oplus A[0] \oplus \cdots \oplus A[-e+1] \end{array}$$

, where $\Phi\left(\underline{\xi}\right) := \begin{pmatrix} \times L & & & & 0 \\ 1_A & \times L[-1] & & & \\ & 1_A[-1] & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1_A[-e+1] & \times L[-e] \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \vdots \\ \xi_e \end{pmatrix}$ *for* $\underline{\xi} \in A \oplus A[-1] \oplus \cdots \oplus A[-e]$.

(3) $\underbrace{\Phi \circ \cdots \circ \Phi}_{i\text{-times}}\left(\underline{\xi}\right) = \Phi^i\left(\underline{\xi}\right) = \begin{pmatrix} \times L^i & & & & 0 \\ \times\left({}_iC_1 \cdot L^{i-1}\right) & \times L^i[-1] & & & \\ \times\left({}_iC_2 \cdot L^{i-2}\right) & \times\left({}_iC_1 \cdot L^{i-1}\right)[-1] & \times L^i[-2] & & \\ \vdots & \ddots & & \ddots & \ddots \\ \times\left({}_iC_e \cdot L^{i-e}\right) & \cdots\ \times\left({}_iC_2 \cdot L^{i-2}\right)[-e+2] & \times\left({}_iC_1 \cdot L^{i-1}\right)[-e+1] & & \times L^i[-e] \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \vdots \\ \xi_e \end{pmatrix}$

for $\underline{\xi} \in A \oplus A[-1] \oplus \cdots \oplus A[-e]$, *where* $L^0 := 1$ *and* $L^{-j} := 0$ *for* $j \in \mathbb{Z}_{\ge 0}$.

(4) $\operatorname{Ker}\Phi^i \subseteq \operatorname{Ker}\left(\times L^i\right) \oplus \operatorname{Ker}\left(\times L^{i+1}[-1]\right) \oplus \cdots \oplus \operatorname{Ker}\left(\times L^{i+e}[-e]\right) = \bigoplus_{j=0}^{e} \operatorname{Ker}\left(\times L^{i+j}[-j]\right)$.

(5) $\operatorname{Ker}\Phi^i\big|_{(\le e-p)} \subseteq \operatorname{Ker}\left(\times L^{i-p}\right) \oplus \operatorname{Ker}\left(\times L^{i+1-p}[-1]\right) \oplus \cdots \oplus \operatorname{Ker}\left(\times L^{i+e-2p}[-e+p]\right) = \bigoplus_{j=0}^{e-p} \operatorname{Ker}\left(\times L^{i+j-p}[-j]\right)$ *for* $p = 0,1,\cdots,e$,

where we denote $\operatorname{Ker}\Phi^i\big|_{(\le j)} := \operatorname{Ker}\Phi^i \cap \left(A \oplus A[-1] \oplus \cdots \oplus A[-j] \oplus 0 \oplus \cdots \oplus 0\right)$ *for* $j = 0,1,\cdots,e$.

(6) *If* $1 \le i \le e$, *then* $\operatorname{Ker}\Phi^i\big|_{(\le e-i)} = 0$.

(7) *If* $\operatorname{Ker}\left(\times L^{i+e-2p}[-(e-p)]\right)_j = 0$ *for all* $p = 0,1,\cdots,e$, *then* $\left(\operatorname{Ker}\Phi^i\right)_j = 0$.

(8) *If* $1 \le i \le e$, *then* $\operatorname{Ker}\left(\times L^{i+e-2p}[-(e-p)]\right)_j = 0$ *for all* $p = 0,1,\cdots,i-1$, *then* $\left(\operatorname{Ker}\Phi^i\right)_j = 0$.

(9) *If* $\operatorname{Ker}\left(\times L^{i+e-2p}[-(e-p)]\right)_j = 0$ *for all* $p = 0,1,\cdots,\min\{e,i\}$, *then* $\left(\operatorname{Ker}\Phi^i\right)_j = 0$.

Proof. (1) follows from Lemma 3.2. (2) follows from the equation $(L+U)\cdot\sum_{j=0}^{e}\xi_j\overline{U^j} = \sum_{j=0}^{e}\left(\xi_{j-1}+L\xi_j\right)\overline{U^j}$. (3) follows from the equation $(M+N)^i = \sum_{j=0}^{i} {}_iC_j M^j N^{i-j}$, where
$$M = \begin{pmatrix} \times L & & & 0 \\ & \times L[-1] & & \\ & & \ddots & \\ 0 & & & \times L[-e] \end{pmatrix},\quad N = \begin{pmatrix} 0 & & & 0 \\ 1_A & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1_A[-e+1] & 0 \end{pmatrix}.$$

(4): If $\underline{\xi} = {}^t\left(\xi_0,\cdots,\xi_e\right) \in \operatorname{Ker}\Phi^i$, where we denote the transpose of the row vector $\left(\xi_0,\cdots,\xi_e\right)$ by ${}^t\left(\xi_0,\cdots,\xi_e\right)$, then we have $\gamma(q) := \sum_{r=0}^{q} {}_iC_{q-r}\cdot L^{i-q+r}\xi_r = 0$ for all $q = 0,1,\cdots,e$ and $L^q\gamma(q) = \sum_{r=0}^{q} {}_iC_{q-r}\cdot L^{i+r}\xi_r = \sum_{r=0}^{q-1} {}_iC_{q-r}\cdot L^{i+r}\xi_r + L^{i+q}\xi_q = 0$.

Since $\left(\gamma(0)=\right)L^i\xi_0 = 0$, we get inductively $L^i\xi_0 = L^{i+1}\xi_1 = \cdots = L^{i+e}\xi_e = 0$ from the above equation. This implies (4).

(5): For given $p(=0,\cdots,e)$, if $\underline{\xi} = {}^t\left(\xi_0,\cdots,\xi_e\right) \in \left(\operatorname{Ker}\Phi^i\right)_j\big|_{(\le e-p)} \subseteq \left(\operatorname{Ker}\Phi^i\right)_j$, then we have $\gamma(q) = \sum_{r=0}^{q} {}_iC_{q-r}\cdot L^{i+r-q}\xi_r = 0$ and $L^{q-p}\gamma(q) = \sum_{r=0}^{q} {}_iC_{q-r}\cdot L^{i+r-p}\xi_r = \sum_{r=0}^{e-p} {}_iC_{q-r}\cdot L^{i+r-p}\xi_r = 0$ for all $q = p,\cdots,e$.

Hence if we put $\underline{\eta} := {}^t\left(L^{i-p}\xi_0, L^{i+1-p}\xi_1,\cdots,L^{i+e-2p}\xi_{e-p}\right)$, then $M^i_{e,p}\underline{\eta} = {}^t\left(L^0\gamma(p),\cdots,L^{e-p}\gamma(p)\right) = 0$. Since $M^i_{e,p}$ is nonsingular matrix by the Lemma 5.3, we have $\underline{\eta} = {}^t\left(L^{i-p}\xi_0, L^{i+1-p}\xi_1,\cdots,L^{i+e-2p}\xi_{e-p}\right) = 0$. This implies (5).

(6): If $\underline{\xi} = {}^t\left(\xi_0,\cdots,\xi_e\right) \in \left(\operatorname{Ker}\Phi^i\right)\big|_{(\le e-i)} \subseteq \operatorname{Ker}\Phi^i$, then $\gamma(q) = \sum_{r=0}^{q} {}_iC_{q-r}\cdot L^{i+r-q}\xi_r = 0$ for all $q = i,\cdots,e$. Hence we have
$\gamma(e) = \sum_{r=0}^{e} {}_iC_{e-r}\cdot L^{i+r-e}\xi_r = \sum_{r=e-i}^{e-i} {}_iC_{e-r}\cdot L^{i+r-e}\xi_r = \xi_{e-i} = 0$, $\gamma(e-1) = \sum_{r=0}^{e-1} {}_iC_{e-1-r}\cdot L^{i+r-e+1}\xi_r = \sum_{r=e-i-1}^{e-i-1} {}_iC_{e-1-r}\cdot L^{i+r-e-1}\xi_r = \xi_{e-i-1} = 0, \cdots,$
$\gamma(i+1) = \sum_{r=0}^{i+1} {}_iC_{i+1-r}\cdot L^{r-1}\xi_r = \sum_{r=1}^{1} {}_iC_{i+1-r}\cdot L^{r-1}\xi_r = \xi_1 = 0$, $\gamma(i) = \sum_{r=0}^{i} {}_iC_{i-r}\cdot L^r\xi_r = \sum_{r=0}^{0} {}_iC_{i-r}\cdot L^r\xi_r = \xi_0 = 0$. This implies (6).

(7): Using (4) and (5) repeatedly, we see that $\operatorname{Ker}\left(\times L^{i+e}[-e]\right)_j = 0$ implies that
$$\left(\operatorname{Ker}\Phi^i\right)_j \subseteq \operatorname{Ker}\Phi^i\big|_{(\le e-1)} \subseteq \operatorname{Ker}\left(\times L^{i-1}\right) \oplus \operatorname{Ker}\left(\times L^i[-1]\right) \oplus \cdots \oplus \operatorname{Ker}\left(\times L^{i+e-2}[-e+1]\right) \oplus 0.$$

So $\operatorname{Ker}\left(\times L^{i+e-2}[-e+1]\right)_j = 0$ implies that
$$\left(\operatorname{Ker}\Phi^i\right)_j \subseteq \operatorname{Ker}\Phi^i\big|_{(\le e-2)} \subseteq \operatorname{Ker}\left(\times L^{i-2}\right) \oplus \operatorname{Ker}\left(\times L^{i-3}[-1]\right) \oplus \cdots \oplus \operatorname{Ker}\left(\times L^{i+e-4}[-e+1]\right) \oplus 0 \oplus 0 \quad \text{and so on.}$$

Finally we get $\left(\operatorname{Ker}\Phi^i\right)_j \subseteq \operatorname{Ker}\Phi^i\big|_{(0)} \subseteq \operatorname{Ker}\left(\times L^{i-e}\right)$ and $\left(\operatorname{Ker}\Phi^i\right)_j \subseteq \operatorname{Ker}\left(\times L^{i-e}\right)_j = 0$.

(8): By the assumption $\mathrm{Ker}\left(\times L^{i+e-2p}[-(e-p)]\right)_j = 0$ for all $p = 0,1,\cdots,i-1$, we see that $\left(\mathrm{Ker}\,\Phi^i\right)_j \subseteq \mathrm{Ker}\,\Phi^i\big|_{(\le e-i)}$ in the similar manner as (7). On the other hand, $\mathrm{Ker}\,\Phi^i\big|_{(\le e-i)} = 0$ by (6). This complete the proof of (8).

(9): This follows from (7) and (8), since $\min\{e,i\} \ge \min\{e,i-1\}$. □

Remark 5.5. If $i+e-2p \ge 0$, then $m(d+e,i)-(e-p) = \left\lfloor \dfrac{d+e-i}{2} - \dfrac{2(e-p)}{2} \right\rfloor = \left\lfloor \dfrac{d-e-i+2p}{2} \right\rfloor = m(d,i+e-2p)$.

Proposition 5.6. *Let i be a positive integer. If $A = A(f)$ has $\Lambda(i,e)$-W.L.P. with respect to $L \in K[\underline{X}]_1$, where $\Lambda(i,e) = \{i+e-2p \mid p = 0,1,\cdots,e\}$, then $B = A_{K[\underline{X},U]}(fu^e)$ has $[i]$-W.L.P. with respect to $L+U \in K[\underline{X},U]_1$.*

Proof. First we remark that $i+e-2p \ge 0$ for all $p = 0,1,\cdots,\min\{e,i\}$. Applying Lemma 5.4 (9) by putting $j = m(d+e,i)$, we have $\mathrm{Ker}\left(\times L^{i+e-2p}[-(e-p)]\right)_{m(d+e,i)} = \mathrm{Ker}\left(\times L^{i+e-2p}\right)_{m(d+e,i)-(e-p)} = \mathrm{Ker}\left(\times L^{i+e-2p}\right)_{m(d,i+e-2p)} = 0$ for all $p = 0,1,\cdots,\min\{e,i\}$ by Remark 5.5 and our assumptions. This implies $\mathrm{Ker}\left(\times(L+U)^i : B_{m(d+e,i)} \to B_{m(d+e,i)+i}\right) = 0$. This complete the proof. □

Theorem 5.7. *Let e,c be positive integers. If $A = A(f)$ has the W.L.P. of order $e+c$ with respect to $L \in K[\underline{X}]_1$, then $B = A_{K[\underline{X},U]}(fu^e)$ has the W.L.P. of order c with respect to $L+U \in K[\underline{X},U]_1$.*

Proof. This follows from Proposition 5.6, since $\Lambda(i,e) = \{i+e-2p \mid p = 0,1,\cdots,e\} \subseteq (-\infty, e+c]$ for all $i = 1,\cdots,c$. □

Corollary 5.8. *If $A = A(f)$ has the S.L.P. with respect to $L \in K[\underline{X}]_1$, then $B = A_{K[\underline{X},U]}(fu^e)$ has the S.L.P. with respect to $L+U \in K[\underline{X},U]_1$.*

Definition 5.9. Let f be a nonzero homogeneous polynomial of $\deg f = d$ in E. The *order of weak Lefschetz property* for f, which is denoted by $\mathrm{ord}_{\mathrm{WLP}} f$, is defined as follows:

$\mathrm{ord}_{\mathrm{WLP}} f = 0$ if and only if $A = A(f)$ does not have the W.L.P.

$1 \le \mathrm{ord}_{\mathrm{WLP}} f = c < d$ if and only if $A = A(f)$ has the W.L.P. of order c and does not has the W.L.P. of order $c+1$.

$\mathrm{ord}_{\mathrm{WLP}} f = \infty$ if and only if $A = A(f)$ has the S.L.P.

Using this terminology, Theorem 5.7 can be interpreted as follows.

Corollary 5.10. $\mathrm{ord}_{\mathrm{WLP}} f - e \le \mathrm{ord}_{\mathrm{WLP}} fu^e$.

Especially if $A = A(f)$ has the S.L.P., then also $A_{K[\underline{X},U]}(fu^e)$ has the S.L.P.

Theorem 5.11. *If $e \ge d \ge 1$, then $B = A_{K[\underline{X},U]}(fu^e)$ has automatically the W.L.P. of order $e-d+1$ with respect to $L+U \in K[\underline{X},U]_1$ for any $L \in K[\underline{X}]_1$.*

Proof. For any $i \le e-d+1$, we have $i \le e$ and $i+e-2p \ge i+e-2(i-1) = e-i+2 > e-i+1 \ge d$ for $p = 0,\cdots,i-1$. So we have $\mathrm{Ker}\left(\times L^{i+e-2p} : 0 = A_{m(d,i+e-2p)} \to A_{m(d,i+e-2p)+i+e-2p}\right) = 0$ for $p = 0,\cdots,i-1$. Applying Lemma 5.4 (8) by putting $j = m(d+e,i)$, we have $\mathrm{Ker}\left(\times L^{i+e-2p}[-(e-p)]\right)_{m(d+e,i)} = \mathrm{Ker}\left(\times L^{i+e-2p}\right)_{m(d+e,i)-(e-p)} = \mathrm{Ker}\left(\times L^{i+e-2p}\right)_{m(d,i+e-2p)} = 0$ for all $p = 0,1,\cdots,i-1$ and all $i \le e-d+1$, here we use Remark 5.5. This implies $\mathrm{Ker}\left(\times (L+U)^i : B_{m(d+e,i)} \to B_{m(d+e,i)+i}\right) = 0$ for all $i \le e-d+1$. We are done. □

Corollary 5.12. *If* $e \ge d \ge 1$ *where* $d = \deg f$ *, then* $\mathrm{ord}_{\mathrm{WLP}} fu^e \ge e-d+1 \ge 1$.

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