Paper

Category of graded modules and Macaulay's inverse system

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In this article, we establish the fundamentals of Macaulay's inverse system by using inner homomorphism of the category of graded modules. As an application, we prove the theorem concerning failure of the weak Lefschetz property for trivial extensions of standard graded Artinian level algebras.

Keywords : Macaulay's inverse system, the weak Lefschetz property, standard graded Artinian level algebras, trivial extensions.

1. Introduction

Macaulay's inverse system appears in many articles, especially related to the Lefschetz properties of Artinian graded algebras, see Ref.(1), Ref.(2), Ref.(3), Ref.(4) and powerful tool for representing Artinian algebras. In fact, any Gorenstein Artinian algebra can be represented by a single element called the Macaulay's inverse generator.

In spite of this, it is scarcely seen the systematic treatment of the fundamentals of Macaulay's inverse system. In this article, we try to do the systematic treatment of the fundamentals of Macaulay's inverse system in case of standard graded Artinian algebras by using inner homomorphism of the category of graded modules. As an application, we prove the theorem concerning failure of the weak Lefschetz property for trivial extensions of standard graded Artinian level algebras, see Theorem 4.24.

In section 1, we establish the fundamentals of Macaulay's inverse system in the category of graded modules. Key results in this section are as follows:

(1) Lemma 2.3, the isomorphism $M^\vee \simeq \hom_{\mathrm{GrMod}(A)}\left(M, A^\vee\right)$ in $\mathrm{GrMod}(A)$, a typical of the change-of-rings isomorphism formula, see A3.13, Ex. A3.50 c, p.677 in Ref.(5).

(2) Lemma 3.4, the isomorphism $E \simeq R^\vee$ in $\mathrm{grMod}(R)$.

(3) Lemma 4.8 (2), $\left\langle 0 \underset{M}{:} \left\langle 0 \underset{M^\vee}{:} N \right\rangle \right\rangle = N$ for $N \overset{\iota}{\hookrightarrow} M$ in $\mathrm{grMod}(A)$, the double annihilator theorem, see Theorem 1.3 in Ref. (6).

(4) Corollary 2.10 (2), $\dim_k M^\vee / \mathfrak{m}_A M^\vee = \dim_k \left(0 \underset{M}{:} \mathfrak{m}_A\right)$ for $M \in \mathrm{GrMod}(A)$.

As a result, any standard graded Artinian algebra admit a Macaulay's inverse system, i.e., any standard graded Artinian algebra can be represented by a Macaulay's inverse system and if the algebra is Gorenstein, then it can be represented by a single element, see Theorem 2.12 and Corollary 2.13. But all results in this section are well known.

In section 3, we recall the result for Macaulay's generator of trivial extensions of standard graded Artinian algebras, see Theorem 4.5. This is also well known but we prove this for the convenience of the reader.

In section 4, we prove the theorem concerning failure of the weak Lefschetz property for trivial extensions of standard graded Artinian level algebras, see Theorem 4.3. This theorem says that even if the level algebra has the strong Lefschetz property, the trivial extension of this algebra may not have the weak Lefschetz property as far as the socle type of this algebra is greater enough than the socle degree and codimension. We remark that similar, but at least approach is quite different, result found in Ref.(6).

2. Graded dual and Macaulay's inverse system

Let K be a field of characteristic 0, $R = R(n) = K\left[X_1, \cdots, X_n\right]$ be a polynomial ring in n variables and R acts on another

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polynomial ring in n variables $E=E(n)=K\left[x_1,\cdots,x_n\right]$ as differential operators defied by $X_i:=\dfrac{\partial}{\partial x_i}\ (i=1,\cdots,n)$. Since K is a field of characteristic 0, E is isomorphic to the injective envelop of the residue field of R in the category of graded R modules.

For any standard graded K algebra of finite type $A=K[A_1]=\bigoplus_{i\in\mathbb{Z}_{\geq 0}} A_i$ with graded maximal ideal $\mathfrak{m}_A:=\bigoplus_{i>0} A_i$, we denote by $\mathrm{GrMod}(A)$ the category of graded A modules with degree preserving morphisms. Let $[i]:\mathrm{GrMod}(A)\to\mathrm{GrMod}(A)$ ($i\in\mathbb{Z}$) be the usual i-shift functor, i.e., $M[i]_j:=M_{i+j}$ for $i,j\in\mathbb{Z}$, $M\in\mathrm{GrMod}(A)$.

Definition 2.1. Given any pair of graded A modules M and N, we define '*inner hom set*' from M to N as follows:

$$\hom_{\mathrm{GrMod}(A)}(M,N)=\hom_{\mathrm{gr}}(M,N):=\bigoplus_{i\in\mathbb{Z}}\mathrm{Hom}_{\mathrm{GrMod}(A)}\left(M[-i],N\right).$$

Especially, since K itself can be seen as a standard graded K algebra concentrated in degree 0, and any graded A module M is also a graded K module, we denote:

$$M^{\vee}:=\hom_{\mathrm{gr}}(M,K)=\bigoplus_{i\in\mathbb{Z}}\mathrm{Hom}_{\mathrm{GrMod}(K)}\left(M[-i],K\right)=\bigoplus_{i\in\mathbb{Z}}\mathrm{Hom}_K\left(M_{-i},K\right)$$

and call $M^{\vee}$ a graded K-dual of M, see A3.4, Ex. A3.5, p.625 in Ref.(5).

Notation 2.2. We denote:

$$N\underset{M}{:}I:=\left\{\xi\in M\,\middle|\,\xi I\subseteq N\right\}(\subseteq M)$$

for $N\subseteq M$ a inclusion of graded A modules and I a subset of A. Similarly we denote:

$$\mathrm{ann}_A(W)=0\underset{A}{:}W:=\left\{\xi\in A\,\middle|\,\xi W=0\right\}(\subseteq A)$$

for a subset W of a graded A module M.

Lemma 2.3. *Let* $M\in\mathrm{GrMod}(A)$. *Then the following hold:*

$$M^{\vee}\simeq\hom_{\mathrm{GrMod}(A)}\left(M,A^{\vee}\right).$$

Proof. Let $\varphi:M^{\vee}=\hom_{\mathrm{GrMod}(K)}(M,K)\to\hom_{\mathrm{GrMod}(A)}\left(M,A^{\vee}\right)$ be a morphism in $\mathrm{GrMod}(A)$ defined by

$$\varphi(f)(\xi)(a):=f(a\xi)\quad\text{for any}\quad f\in M^{\vee},\,\xi\in M,\,a\in A.$$

Let $\psi:\hom_{\mathrm{GrMod}(A)}\left(M,A^{\vee}\right)\to M^{\vee}=\hom_{\mathrm{GrMod}(K)}(M,K)$ be a morphism in $\mathrm{GrMod}(A)$ defined by

$$\psi(g)(\xi):=g(\xi)(1)\quad\text{for any}\quad g\in\hom_{\mathrm{GrMod}(A)}\left(M,A^{\vee}\right),\,\xi\in M.$$

Then φ and ψ are mutually inverses. □

Lemma 2.4. $E\simeq R^{\vee}$ in $\mathrm{grMod}(R)$.

Proof. We can construct an isomorphism in $\mathrm{grMod}(R)$ as follows:

Let $\theta : E \to R^{\vee}$ be a morphism defined by $\theta(f)(\xi) := (\xi f)_0 = (\text{constant term of } \xi f \in E)$ for any $f \in E, \xi \in R$, then θ is clearly a morphism in $\mathrm{grMod}(R)$ and also an isomorphism. In fact, non-zero polynomial $0 \neq f \in E$ can be expressed in the following form:

$$f = c x_1^{e_1} \cdots x_n^{e_1} + \cdots$$

where $c \in K$ is the non-zero coefficient of a monomial $x_1^{e_1} \cdots x_n^{e_1}$ appearing in f with $\deg f = e_1 + \cdots + e_n$. So $\theta(f)\left(X_1^{e_1} \cdots X_n^{e_1}\right) = e_1! \cdots e_n! c \neq 0$. This implies θ is injective. Comparing dimension of each homogeneous component of E and $R^{\vee}$, we can deduce that θ is also surjective, hence isomorphism. □

Remark 2.5. The following hold:

(1) $\hom_{\mathrm{gr}}(M, N)$ has the natural graded A module structure, so $\hom_{\mathrm{gr}}(M, N) \in \mathrm{GrMod}(A)$.

(2) Let $\mathrm{grMod}(A)$ denote the full subcategory of $\mathrm{GrMod}(A)$, whose objects are degree-wise finite dimensional modules, i.e., $M \in \mathrm{grMod}(A)$ if and only if $M \in \mathrm{GrMod}(A)$ and $\dim_K M_i < \infty$ for all $i \in \mathbb{Z}$. Restricting to $\mathrm{grMod}(A)$, K-dual functor is a contravariant exact idempotent K-linear functor :

$$(\_)^{\vee} : \mathrm{grMod}(A) \to \mathrm{grMod}(A).$$

(3) $A^{\vee}$ is an injective object in $\mathrm{grMod}(A)$, since we have the following commutative diagram with exact rows for any $0 \to N \to M$ (exact) in $\mathrm{grMod}(A)$:

$$\begin{array}{ccccc} N^{\vee} & \to & M^{\vee} & \to & 0 \\ \| & \circlearrowright & \| & & \\ \hom_{\mathrm{GrMod}(A)}(N, A^{\vee}) & \to & \hom_{\mathrm{GrMod}(A)}(M, A^{\vee}) & & \end{array}.$$

Especially $R^{\vee}$ is an injective object in $\mathrm{grMod}(R)$.

(4) $E \simeq R^{\vee}$ are isomorphic to $E_R^{\mathrm{gr}}(k)$ the injective hull of the residue field k in $\mathrm{grMod}(R)$, since we can easily check that the following extension

$$k = R / \mathfrak{m}_R \simeq 0 \underset{E}{:} \mathfrak{m}_R = K \cdot 1 \hookrightarrow E$$

is an essential extension of the residue field k of R and $E \simeq R^{\vee}$ is an graded injective module in $\mathrm{grMod}(R)$ by (3).

(5) If $\phi : R \twoheadrightarrow A$ is a surjective K-algebra homomorphism, then we have the following commutative diagram with injective morphism $\phi^{\vee}$, inclusions and isomorphisms :

$$\begin{array}{ccccc} A^{\vee} & \xrightarrow{\phi^{\vee}} & R^{\vee} & \simeq & E \\ \cup & \circlearrowright & \cup & \circlearrowright & \cup \\ k \simeq 0 \underset{A^{\vee}}{:} \mathfrak{m}_A & \simeq & 0 \underset{R^{\vee}}{:} \mathfrak{m}_R & \simeq & 0 \underset{E}{:} \mathfrak{m}_R = K \cdot 1 \end{array}.$$

This implies $k = A / \mathfrak{m}_A \simeq 0 \underset{A^{\vee}}{:} \mathfrak{m}_A \hookrightarrow A^{\vee}$ is an essential extension of the residue field k of A. So $A^{\vee}$ is isomorphic to $E_A^{\mathrm{gr}}(k)$ the graded injective hull of the residue field k in $\mathrm{grMod}(A)$. If A is an Artinian ring, then $A^{\vee}$ is also isomorphic to the graded canonical module K_A of A, since $A^{\vee}$ is finitely generated graded injective A module of type one, i.e, $\dim_k 0 \underset{A^{\vee}}{:} \mathfrak{m}_A = 1$.

Notation 2.6. We denote:

$$\left\langle 0 \underset{M^\vee}{:} N \right\rangle := \left\{ f \in M^\vee \middle| f(N) = 0 \right\} (\in \mathrm{GrMod}(A)).$$

for $N \subseteq M$ a inclusion of graded A modules. Similarly we denote:

$$\left\langle 0 \underset{M}{:} W \right\rangle := \left\{ \xi \in M \middle| f(\xi) = 0, \text{ for all } f \in W \right\} (\in \mathrm{GrMod}(A))$$

for $W \subseteq M^\vee$ a inclusion of graded A modules.

Lemma 2.7. *If* $M \in \mathrm{grMod}(A)$*, then we have the following commutative diagram:*

$$\begin{array}{ccc} \left\langle 0 \underset{M}{:} W \right\rangle & \hookrightarrow & M \\ \| & \circlearrowleft & \| \\ \left\langle 0 \underset{(M^\vee)^\vee}{:} W \right\rangle & \hookrightarrow & (M^\vee)^\vee \end{array}$$

for any graded submodule $W \subseteq M^\vee$*. Especially* $\left\langle 0 \underset{M}{:} W \right\rangle \simeq \left\langle 0 \underset{(M^\vee)^\vee}{:} W \right\rangle$.

Proof. The assertion follows straightforward, since $(M^\vee)^\vee$ is naturally isomorphic to M. □

Lemma 2.8. *Let* $N \overset{\iota}{\hookrightarrow} M$ *be a inclusion of graded* A *modules. Then the following hold:*

(1) $\mathrm{Ker}(\iota^\vee) = \left\langle 0 \underset{M^\vee}{:} N \right\rangle \simeq (M/N)^\vee$.

Moreover if $M \in \mathrm{grMod}(A)$*, then the following hold:*

(2) $\left\langle 0 \underset{M}{:} \left\langle 0 \underset{M^\vee}{:} N \right\rangle \right\rangle = N$.

Proof. (1): We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & (M/N)^\vee & \to & M^\vee & \overset{\iota^\vee}{\to} & N^\vee & \to & 0 \\ & & \| & \circlearrowleft & \| & & & & \\ & & \left\langle 0 \underset{M^\vee}{:} N \right\rangle & \overset{\zeta}{\hookrightarrow} & M^\vee & & & & \end{array}.$$

This implies $\left\langle 0 \underset{M^\vee}{:} N \right\rangle \simeq (M/N)^\vee$. (2): Moreover if $M \in \mathrm{grMod}(A)$, then by taking graded K -dual of the above diagram and applying Lemma 2.7, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \left\langle 0 \underset{M}{:} W \right\rangle & \hookrightarrow & M & \to & M/N & \to & 0 \\ & & \| & \circlearrowleft & \| & \circlearrowleft & \| & & \\ 0 & \to & \left\langle 0 \underset{(M^\vee)^\vee}{:} W \right\rangle & \hookrightarrow & (M^\vee)^\vee & \overset{\zeta^\vee}{\to} & W^\vee & \to & 0 \end{array}$$

, where $W = \left\langle 0 \underset{M^\vee}{:} N \right\rangle$. This implies $\left\langle 0 \underset{M}{:} \left\langle 0 \underset{M^\vee}{:} N \right\rangle \right\rangle = N$. □

Remark 2.9. The following hold:

(1) $\left\langle 0 \underset{M^\vee}{:} IM \right\rangle = 0 \underset{M^\vee}{:} I$ if $M \in \mathrm{GrMod}(A)$ and $I \subseteq A$ is a homogeneous ideal.

(2) $\left\langle 0 \underset{A}{:} M \right\rangle = 0 \underset{A}{:} M$ if $M \subseteq A^\vee$ is a graded A submodule.

Corollary 2.10. *If* $M \in \mathrm{GrMod}(A)$, *then the following hold:*

(1) $(M / \mathfrak{m}_A M)^\vee \simeq 0 \underset{M^\vee}{:} \mathfrak{m}_A$ $\quad \dim_K (M / \mathfrak{m}_A M)_i = \dim_K \left(0 \underset{M^\vee}{:} \mathfrak{m}_A\right)_{-i}$ *for any* $i \in \mathbb{Z}$.

Especially $\dim_k M / \mathfrak{m}_A M = \dim_k \left(0 \underset{M^\vee}{:} \mathfrak{m}_A\right)$.

Moreover if $M \in \mathrm{grMod}(A)$, *then the following hold:*

(2) $\left(M^\vee / \mathfrak{m}_A M^\vee\right)^\vee \simeq 0 \underset{M}{:} \mathfrak{m}_A$ $\quad \dim_k \left(M^\vee / \mathfrak{m}_A M^\vee\right)_{-i} = \dim_k \left(0 \underset{M}{:} \mathfrak{m}_A\right)_i$ *for any* $i \in \mathbb{Z}$.

Especially $\dim_k M^\vee / \mathfrak{m}_A M^\vee = \dim_k \left(0 \underset{M}{:} \mathfrak{m}_A\right)$.

Proof. (1) : From Lemma 2.8 (1) and Remark 2.9 (1), $(M / \mathfrak{m}_A M)^\vee \simeq \left\langle 0 \underset{M^\vee}{:} \mathfrak{m}_A M \right\rangle \simeq 0 \underset{M^\vee}{:} \mathfrak{m}_A$.

(2) : By (1), we have $\left(M^\vee / \mathfrak{m}_A M^\vee\right)^\vee \simeq 0 \underset{(M^\vee)^\vee}{:} \mathfrak{m}_A \simeq 0 \underset{M}{:} \mathfrak{m}_A$ since $\left(M^\vee\right)^\vee \simeq M$. □

Remark 2.11. From Corollary 2.10 (2), we have $\dim_k A^\vee / \mathfrak{m}_A A^\vee = \dim_k \left(0 \underset{A}{:} \mathfrak{m}_A\right)$.

The following Theorem 2.12 and Corollary 2.13 are well known, see Theorem 2.1 in Ref.(4) or Ref.(7). But using Lemma 2.8, Remark 2.5 (4) , Remark 2.9 and Remark 2.11, we can prove these easily.

Theorem 2.12.(Macaulay's inverse system) *Let* A *be a standard* $\mathbb{Z}$*-graded* K *algebra of embedding dimension* n *and* $\phi : R \twoheadrightarrow A$ *be a surjective* K *algebra homomorphism. Then* $A^\vee \simeq 0 \underset{E}{:} \mathrm{Ker}\phi$ *as graded* R *modules, where we assume* $A^\vee$ *is an graded* R *module via* ϕ, *and* $0 \underset{R}{:} \left(0 \underset{E}{:} \mathrm{Ker}\phi\right) = \mathrm{Ker}\phi$. *In this case, we call the graded* R *submodule* $0 \underset{E}{:} I$ *of* E *the inverse system of the homogeneous ideal* $I := \mathrm{Ker}\phi \subseteq R$.

Proof. From Lemma 2.8 (1) and Remark 2.5 (4), $A^\vee \simeq (R / \mathrm{Ker}\phi)^\vee \simeq 0 \underset{R^\vee}{:} \mathrm{Ker}\phi \simeq 0 \underset{E}{:} \mathrm{Ker}\phi$. Since $0 \underset{R^\vee}{:} \mathrm{Ker}\phi \simeq 0 \underset{E}{:} \mathrm{Ker}\phi$, we have $0 \underset{R}{:} \left(0 \underset{E}{:} \mathrm{Ker}\phi\right) = 0 \underset{R}{:} \left(0 \underset{R^\vee}{:} \mathrm{Ker}\phi\right)$.

From Lemma 2.8 (2) and Remark 2.9, we have $0 \underset{R}{:} \left(0 \underset{E}{:} \mathrm{Ker}\phi\right) = 0 \underset{R}{:} \left(0 \underset{R^\vee}{:} \mathrm{Ker}\phi\right) = \left\langle 0 \underset{R}{:} \left\langle 0 \underset{R^\vee}{:} \mathrm{Ker}\phi \right\rangle \right\rangle = \mathrm{Ker}\phi$. □

Corollary 2.13. *If* A *is a standard graded algebra of embedding dimension* n *then there exists a graded* R *submodule* $M \subseteq E$ *such that* $A \simeq R / I(M)$ *where* $I(M) := 0 \underset{R}{:} M$ *and* $A^{\vee} \simeq M$. *Moreover the following hold:*

(1) M *is a non-zero finitely generated* R *module if and only if* A *is Artinian.*

(2) *If* $A = \bigoplus_{i=0}^{d} A_i$ *is a standard graded Gorenstein Artinian algebra with* $A_d \neq 0$, *then* $A^{\vee} \simeq M$ *is a single generated graded* R *submodule and there exists a homogeneous polynomial* f *in* E *of degree* d *such that* $M = Rf$ *Especially,* $A^{\vee}[-d] \simeq A$.

Proof. (1) Since $M \simeq A^{\vee}$ is an injective A module by Remark 2.5 (3) and A is a Noetherian ring, if M is finitely generated then $\text{Kull-dim}\, A = \text{inj-dim}_A M = 0$. Hence A is an Artinian ring. The invers statement is clear.

(2) If A is a standard graded Gorenstein Artinian algebra, then M is single generated since $\dim_k M / \mathfrak{m}_A M = \dim_k A^{\vee} / \mathfrak{m}_A A^{\vee} = \dim_k \left(0 \underset{A}{:} \mathfrak{m}_A \right) = 1$ by Remark 2.11. Moreover we have

$$\left(0 \underset{A}{:} \mathfrak{m}_A \right) = \left(0 \underset{A}{:} \mathfrak{m}_A \right)_d \simeq \left(M / \mathfrak{m}_A M \right)_{-d} = M / \mathfrak{m}_A M .$$

Hence $A \simeq R / I(M) \simeq R f[-d] = M[-d] \simeq A^{\vee}[-d]$ for some homogeneous polynomial $f \in M_{-d} \subseteq E$ of degree d.

Here we remark that the degree of a homogeneous element f in E as a graded R module is negative of the degree of f as a polynomial in E. □

Proposition 2.14. (Minimal injective resolution) *Let* $A \simeq R / I$ *be a standard* $\mathbb{Z}$*-graded Artinian algebra of embedding dimension* n, M *be a finitely generated graded* A *module and* F *be a minimal free resolution of* M *over* R. *Then* $F^{\vee}$ *is a minimal injective resolution of finitely generated module* $M^{\vee}$ *over* R.

Proof. From Remark 1.5 (2), (3), $F^{\vee}$ is an injective resolution of $M^{\vee}$ over R.

Since $\mathrm{RHom}_R(M,k) \simeq \mathrm{RHom}_R(k, M^{\vee})$, where $\mathrm{RHom}_R(\ ,\ )$ denote the right derived hom functor, and Corollary 2.10 (a), we have

$$\dim_k \left(0 \underset{(F_i)^{\vee}}{:} \mathfrak{m}_R \right) = \dim_k F_i / \mathfrak{m} F_i = \dim_k \mathrm{Ext}_R^i(M,k) = \dim_k \mathrm{Ext}_R^i(k, M^{\vee}) .$$

So $F^{\vee}$ is also a minimal one. □

Notation 2.15. Let $f_1, \cdots, f_r$ be homogeneous polynomials in E. We denote:

$$A^{\vee}(f_1, \cdots, f_r) := \sum_{i=1}^{r} R f_i \subseteq E ,$$

$$I(f_1, \cdots, f_r) := 0 \underset{R}{:} A^{\vee}(f_1, \cdots, f_r) = 0 \underset{R}{:} \{f_1, \cdots, f_r\} ,$$

$$A(f_1, \cdots, f_r) := R / I(f_1, \cdots, f_r) = R \Big/ \bigcap_{i=1}^{r} \mathrm{ann}_R(f_i) .$$

Especially, $A = A(f_1, \cdots, f_r)$ is an Artinian algebra with $\text{emb.dim}\, A \leq \text{emb.dim}\, R = n$ and $A(f_1, \cdots, f_r)^{\vee} \simeq A^{\vee}(f_1, \cdots, f_r)$.

Definition 2.16. Let $M \in \mathrm{grMod}(R)$ be an graded R module whose each component is finite dimensional. We define the Hilbert function $h_M : \mathbb{Z} \to \mathbb{Z}$ of M by $h_M(i) = \dim_K M_i$ for $i \in \mathbb{Z}$.

3. Trivial extension of standard graded Artinian algebras and its Macauly's generator

In this section, we review the result for Macaulay's generator of trivial extensions of standard graded Artinian algebras.

Definition 3.1. Let A be a standard graded K algebra of finite type and V be a finitely generated graded A module. We define a K algebra of finite type $A \ltimes V$ called the '*trivial extension*' (or '*Nagata idealization*') of A by V as follows:

(1) $A \ltimes V := A \oplus V$ as an A module;

(2) $(a,v)\cdot(b,w) := (ab, aw+bv)$ for any $(a,v),(b,w) \in A \ltimes V = A \oplus V$.

Definition 3.2. Let $A = \bigoplus_{i=0}^{d} A_i$ be an Artinian standard graded K algebra with $A_d \neq 0$. A is called a '*level algebra*', if $0 \underset{A}{:} \mathfrak{m}_A = \left(0 \underset{A}{:} \mathfrak{m}_A\right)_d$. The number d is called the socle degree of A and $t = \dim_K\left(0 \underset{A}{:} \mathfrak{m}_A\right)$ is called the type of A.

Remark 3.3. We remark the following:

(1) Assume that A and B are both graded K algebras of finite type and that B is an A algebra with its structure morphism $\varphi : A \to B$. If there exist $V \subseteq B$ a graded A submodule with $V^2 = 0$, $\varphi(A) \simeq A$ and $B \simeq \varphi(A) \oplus V$, then $B \simeq A \ltimes V$ as graded K algebras.

(2) Trivial extensions of standard graded K algebras by their graded modules are graded but no longer standard graded in general.

(3) If A is a standard graded Artinian level K algebra with its canonical module K_A, then $A \ltimes K_A$ is a standard graded Gorenstein K algebra.

Lemma 3.4. *Let $f_1,\cdots,f_r \in E(\underline{x})$ be nonzero homogeneous polynomials, $A := A_{K[\underline{X}]}(f_1,\cdots,f_r)$ and $g := u_1 f_1 + \cdots + u_r f_r \in E(\underline{x},\underline{u})$. Then the following hold:*

(1) $A_{K[\underline{X},\underline{U}]}(g) = A\cdot 1 + \sum_{i=1}^{r} A\cdot\overline{U_i}$, *where* $\overline{U_i}$ *is the homomorphic image of* U_i *in* $A_{K[\underline{X},\underline{U}]}(g)$ *for* $i = 1,\cdots,r$.

(2) $A \simeq A\cdot g = K[\underline{X}]\cdot g \subseteq E(\underline{x},\underline{u})$, $K_A \simeq A^\vee = V\cdot g \subseteq E(\underline{x},\underline{u})$ *and* $A\cdot g \cap V\cdot g = 0$ *in* $E(\underline{x},\underline{u})$, *where* $V := \sum_{i=1}^{r} A\cdot\overline{U_i}$.

Proof. Since $A = K[\underline{X}]/I_{K[\underline{X}]}(f_1,\cdots,f_r)$ and $I_{K[\underline{X}]}(f_1,\cdots,f), \{U_i U_j\}_{1\le i\le j\le r} \subseteq I_{K[\underline{X},\underline{U}]}(g)$, we have

$$A_{K[\underline{X},\underline{U}]}(g) = K[\underline{X},\underline{U}]/I(g) = A\cdot 1 + \sum_{i=1}^{r} A\cdot\overline{U_i}.$$

Since $0 \underset{K[\underline{X}]}{:} g = 0 \underset{K[\underline{X}]}{:} f_1\cdot u_1 + \cdots + f_r\cdot u_r = \bigcap_{i=1}^{r} \mathrm{ann}_{K[\underline{X}]}(f_i) = I_{K[\underline{X}]}(f_1,\cdots,f)$, we have

$$A = K[\underline{X}]/I_{K[\underline{X}]}(f_1,\cdots,f) \simeq K[\underline{X}]\cdot g = A\cdot g.$$

We remark that $V \cdot g = \left(\sum_{i=1}^{r} A \cdot \overline{U_i} \right) \cdot \left(f_1 \cdot u_1 + \cdots + f_r \cdot u_r \right) = \sum_{i=1}^{r} A \cdot f_i = A^{\vee} \simeq K_A$. Finally $A \cdot g \subseteq E(\underline{x}) \cdot u_1 + \cdots + E(\underline{x}) \cdot u_r$ and $V \cdot g = \sum_{i=1}^{r} A \cdot f_i \subseteq E(\underline{x})$. This implies $A \cdot g \cap V \cdot g = 0$ in $E(\underline{x}, \underline{u})$. □

The following Theorem 3.5 is well known but we give a proof by using Lemma 3.4.

Theorem 3.5.(p.82. Theorem 2.77 in Ref.(1)) *Let* $f_1, \cdots, f_r$ *be nonzero homogeneous polynomials in* $E(\underline{x})$. *Then*

$$A_{K[\underline{X},\underline{U}]}\left(f_1 \cdot u_1 + \cdots + f_r \cdot u_r \right) \simeq A_{K[\underline{X}]}\left(f_1, \cdots, f_r \right) \ltimes K_{A(f_1, \cdots, f_r)}.$$

Proof. By Lemma 3.4. (1), $A_{K[\underline{X},\underline{U}]}(g) = A \cdot 1 + V$ where $g := f_1 \cdot u_1 + \cdots + f_r \cdot u_r \in E(\underline{x}, \underline{u})$, $V := \sum_{i=1}^{r} A \cdot \overline{U_i}$ with $\overline{U_i}$ the homomorphic image of U_i in $A_{K[\underline{X},\underline{U}]}(g)$ for $i = 1, \cdots, r$. Clearly $V^2 = 0$.

The isomorphism $A_{K[\underline{X},\underline{U}]}(g) = A \cdot 1 + V \to A_{K[\underline{X},\underline{U}]}(g) \cdot g = A \cdot g + V \cdot g$ induces $A_{K[\underline{X},\underline{U}]}(g) = A \cdot 1 + V \simeq A \oplus V$ and $V \simeq K_A$, since by Lemma 3.4. (2), $A \cdot g \simeq A$, $V \cdot g \simeq A^{\vee} \simeq K_A$ and $A \cdot g \cap V \cdot g = 0$. This completes the proof. □

4. Failure of the weak Lefschetz property for trivial extensions of standard graded Artinian level algebras

First we recall the definition of the weak Lefschetz property (WLP) of a standard graded Artinian K algebra.

Definition 4.1. Let A be a standard graded K algebra. Given a nonzero homogeneous element $\alpha \in A$ and a graded A module M whose components are all finite dimensional, we say that $\times\alpha|_M : M \to M[\deg\alpha]$ has the '*maximal rank*' property if each component $\left(\times\alpha|_M \right)_i : M_i \to M_{i+\deg\alpha}$ ($i \in \mathbb{Z}$) has maximal rank, i.e. $\dim_K \left(\times\alpha|_M (M_i) \right) = \min\left\{ \dim_K M_i, \dim_K M_{i+\deg\alpha} \right\}$ for all $i \in \mathbb{Z}$.

Definition 4.2. Let A be a standard graded Artinian K algebra. We say that A has the '*weak Lefschetz property'* (WLP) if there exists a nonzero linear form $z \in A_1$ such that $\times z|_A : A \to A[1]$ has the maximal rank property. In this case, we say that A has WLP with respect to z. Moreover if A is a homomorphic image of a polynomial ring $K[\underline{X}]$, i.e., $A \simeq K[\underline{X}]/I$, then A is a graded $K[\underline{X}]$-module. If $L \in K[\underline{X}]_1$ is a preimage of $z \in A_1$, then we also say that A has WLP with respect to $L \in K[\underline{X}]_1$. We remark that $A^{\vee}(f)[-d] \simeq A(f)$ with $\deg f = d$, so $A(f)$ has the symmetric Hilbert function, i.e., $h_A(i) = h_A(d-i)$ for all $i \in \mathbb{Z}$.

Theorem 4.3. *Let* $f_1, \cdots, f_r$ *be nonzero homogeneous polynomials of the same degree* $d \geq 2$ *in* $K[x_1, \cdots, x_n] = E(\underline{x})$ *and* $g := f_1 \cdot u_1 + \cdots + f_r \cdot u_r$ *be a polynomial of degree* $d+1$ *in* $K[x_1, \cdots, x_n, u_1, \cdots, u_r] = E(\underline{x}, \underline{u})$, *which satisfy the following conditions:*

(1) $\dim_K \left(K f_1 + \cdots + K f_r \right) = r$, *i.e.,* $f_1, \cdots, f_r$ *are linearly independent;*

(2) $\dim_K \left(K \frac{\partial g}{\partial x_1} + \cdots + K \frac{\partial g}{\partial x_n} \right) = n$ *i.e.,* $\frac{\partial g}{\partial x_1}, \cdots, \frac{\partial g}{\partial x_n}$ *are linearly independent;*

(3) $\dim_K A(f_1, \cdots, f_r)_d > \dim_K A(f_1, \cdots, f_r)_{d-1}$, *i.e.,* $\dim_K A^{\vee}(f_1, \cdots, f_r)_{-d} > \dim_K A^{\vee}(f_1, \cdots, f_r)_{-d+1}$,

then $A_{K[\underline{X},\underline{U}]}\left(f_1\cdot u_1+\cdots+f_r\cdot u_r\right)=A(g)$ *does not have the the weak Lefschetz property.*

Proof. By Lemma 3.22. in Ref.(1), it is enough to show that $\dim_K A(g)_1>\dim_K A(Lg)_1$, *i.e.*,

$$\dim_K A^{\vee}(g)_{-d}>\dim_K A^{\vee}(Lg)_{-d+1}$$

for general linear element $L=a_1X_1+\cdots+a_nX_n+b_1U_1+\cdots+b_rU_r\in K\left[X_1,\cdots,X_n,U_1,\cdots,U_r\right]_1$.

Frist we remark that $A^{\vee}(g)_{-d}=\sum_{i=1}^{n}K\frac{\partial g}{\partial x_i}+\sum_{j=1}^{r}K\frac{\partial g}{\partial u_j}=\sum_{i=1}^{n}K\frac{\partial g}{\partial x_i}+\sum_{j=1}^{r}Kf_j$ and $\sum_{i=1}^{n}K\frac{\partial g}{\partial x_i}\cap\sum_{j=1}^{r}Kf_j=0$ in $E(\underline{x},\underline{u})$ since $\sum_{i=1}^{n}K\frac{\partial g}{\partial x_i}\subseteq(u_1,\cdots,u_n)\cdot K\left[x_1,\cdots,x_n,u_1,\cdots,u_r\right]$ and $\sum_{j=1}^{r}Kf_j\subseteq K\left[x_1,\cdots,x_n\right]\subseteq K\left[x_1,\cdots,x_n,u_1,\cdots,u_r\right]$. By the conditons (1) and (2), we have $\dim_K A^{\vee}(g)_{-d}=n+r$. On the other hand, $A^{\vee}(Lg)_{-d+1}=\sum_{i=1}^{n}K\frac{\partial Lg}{\partial x_i}+\sum_{j=1}^{r}K\frac{\partial Lg}{\partial u_j}=\sum_{i=1}^{n}K\frac{\partial Lg}{\partial x_i}+\sum_{j=1}^{r}K\left(\sum_{i=1}^{n}a_i\frac{\partial f_j}{\partial x_i}\right)$ since $Lg=\sum_{i=1}^{n}\sum_{j=1}^{r}a_i\frac{\partial f_j}{\partial x_i}u_j+\sum_{j=1}^{r}b_jf_j$ and $\frac{\partial Lg}{\partial u_j}=\sum_{i=1}^{n}a_i\frac{\partial f_j}{\partial x_i}$ $(j=1,\cdots,r)$. By the conditons (1) and (3),

$\dim_K\sum_{j=1}^{r}K\left(\sum_{i=1}^{n}a_i\frac{\partial f_j}{\partial x_i}\right)\leq\dim_K A^{\vee}(f_1,\cdots,f_r)_{-d+1}<\dim_K A^{\vee}(f_1,\cdots,f_r)_{-d}=r$, so we have $\dim_K A^{\vee}(Lg)_{-d+1}<n+r$. This proves the assertion. □

Lemma 4.4. *Let* $f_1,\cdots,f_r$ *be nonzero homogeneous polynomials of the same degree* $d\geq 2$ *in* $K\left[x_1,\cdots,x_n\right]=E(\underline{x})$ *and* $g:=f_1\cdot u_1+\cdots+f_r\cdot u_r$ *be a polynomial of degree* $d+1$ *in* $K\left[x_1,\cdots,x_n,u_1,\cdots,u_r\right]=E(\underline{x},\underline{u})$. *If there exists* $f\in\{f_1,\cdots,f_r\}$ *such that* $\dim_K\left(K\frac{\partial f}{\partial x_1}+\cdots+K\frac{\partial f}{\partial x_n}\right)=n$, *then the condition* (2) $\dim_K\left(K\frac{\partial g}{\partial x_1}+\cdots+K\frac{\partial g}{\partial x_n}\right)=n$ *in Theorem 4.3 holds.*

Proof. We can assume that $f=f_1$. If $c_1\frac{\partial g}{\partial x_1}+\cdots+c_n\frac{\partial g}{\partial x_n}=c_1\sum_{j=1}^{r}\frac{\partial f_j}{\partial x_1}u_j+\cdots+c_n\sum_{j=1}^{r}\frac{\partial f_j}{\partial x_n}u_j=0$ for some $c_1,\cdots,c_n\in K$, then substituting $u_2=\cdots=u_n=0$, we have $\left(c_1\frac{\partial f_1}{\partial x_1}+\cdots+c_n\frac{\partial f_1}{\partial x_n}\right)u_1=0$. Hence $c_1=\cdots=c_n=0$ by the assumption. This complete the proof. □

Example 4.5. Let $f_1=x_1^2,\ f_2=x_1x_2,\ f_3=x_2^2\in K\left[x_1,x_2\right]$, then the standard graded Artinian level algebra $A=A(f_1,f_2,f_3)$ of socle type 3 has the strong Lefschetz property since it is well known that every Artinian algebra of codimension 2 has the strong Lefschetz property. Hence A has also the weak Lefschetz property. In spite of this, $A(g)$ the trivial extension of A, where $g=f_1u_1+f_2u_2+f_3u_r\in K[x_1,x_2,u_1,u_2,u_3]$, does not have the the weak Lefschetz property since the following three conditions in Theorem 4.3 holds:

(1) $\dim_K\left(K\,f_1+K\,f_2+K\,f_3\right)=3$.

(2) Since $\dim_K\left(K\frac{\partial f_2}{\partial x_1}+K\frac{\partial f_2}{\partial x_2}\right)=\dim_K\left(K\,x_2+K\,x_1\right)=2$, by Lemma 3.25, we have $\dim_K\left(K\frac{\partial g}{\partial x_1}+K\frac{\partial g}{\partial x_2}\right)=2$.

(3) $\dim_K A(f_1,f_2,f_3)_{-2}=\dim_K\left(K\,f_1+K\,f_2+K\,f_3\right)=3>\dim_K A(f_1,f_2,f_r)_{-1}=\dim_K\left(K\,x_2+K\,x_1\right)=2$.

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