Paper

Strongly $\mathfrak{m}$-full modules

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Abstract We introduce the stongly $\mathfrak{m}$ - full property for a pair of graded modules, which is stronger than the $\mathfrak{m}$ - full property but weaker than the componentwise $\mathfrak{m}$ - full property, and give criteria for strong $\mathfrak{m}$ - fullness by means of strongly $\mathfrak{m}$ - full closure. One more property for a pair of graded modules : the $\mathfrak{m}$ - adic $\mathfrak{m}$ - fullness is introduced. Using strongly $\mathfrak{m}$ - full closure , we show that $\mathfrak{m}$ - adic $\mathfrak{m}$ - fullness and componentwise $\mathfrak{m}$ - fullness are equivalent.

Keywords : Standard graded commutative algebra, Strongly $\mathfrak{m}$ - full modules, $\mathfrak{m}$ - acically $\mathfrak{m}$ - full modules, Componentwise $\mathfrak{m}$ - full modules.

1. Intoroduction

The property of homogeneous ideals of a standard graded Noetherian commutative algebra over a field, called the $\mathfrak{m}$-fullness, and related topics have been studied by many authors (e.g., (1)-(13)). The property, $\mathfrak{m}$-fullness, can naturally be extended to the property of a pair of a graded module and its submodule.

In this paper, we introduce the strongly $\mathfrak{m}$-full property, which is stronger than the $\mathfrak{m}$-full property, for a pair of a graded module and its graded submodule, but weaker than componentwise $\mathfrak{m}$-full property introduced in (8). We also introduce the $\mathfrak{m}$-adically $\mathfrak{m}$-full property which is equivalent to componentwise $\mathfrak{m}$-full property. We give criteria for those properties.

In Section 2, the preliminary section, we fix some notations used in (8) and recall the criteria for $\mathfrak{m}$-fullness and the definition of componentwise $\mathfrak{m}$-full defect from (8).

In Section 3, we study strongly $\mathfrak{m}$-fullness. Introducing the strongly $\mathfrak{m}$-full closure, we give criteria for strongly $\mathfrak{m}$-fullness in Corollary 3.11.

In Section 4, we define $\mathfrak{m}$-adically $\mathfrak{m}$-full property and after showing that this property is equivalent to componentwise $\mathfrak{m}$-fullness, give criteria for those equivalent properties in Theorem 4.9.

In Section 5, the last section, applying our results to homogeneous ideals of the polynomial ring in two variables, we extend the result in (11, Theorem 4) as Theorem 5.11.

2. Preliminaries

Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}_{\geq 0}$ be the set of nonnegative integers and R be a standard graded Noetherian commutative algebra over an infinite field K with the maximal homogeneous ideal $\mathfrak{m}$ and the residue field $k \simeq R/\mathfrak{m}$. Let $\mathcal{A}$ be the category of finitely generated graded R-modules, for $M \in \mathcal{A}$ and $i \in \mathbb{Z}$, we denote M_i the K-vector subspace generated by homogeneous elements of degree i in M, $M(i)$ the i-shifted module of M defined by $(M(i))_j := i + j$ for all $j \in \mathbb{Z}$. For a graded submodule N of M and a homogeneous ideal I of R, we denote $N \underset{M}{:} I := \{\xi \in M \mid I\xi \subseteq N\}$, and similarly, for a homogeneous element $z \in R$, we also denote $N \underset{M}{:} z := \{\xi \in M \mid z\xi \subseteq N\}$. We fix some notations as follows:

Notation 2.1. Let $N \subseteq M \in \mathcal{A}$ and $j \in \mathbb{Z}$.

(1) $M_{\geq j} := \bigoplus_{i \geq j} M_i$: the graded submodule of elements of degrees greater than or equal to j in M.

(2) $M_{\langle j \rangle} := RM_j$: the graded submodule generated by elements of degrees j in M.

(3) $\deg : \coprod_{i \in \mathbb{Z}} (M_i \setminus \{0\}) \to \mathbb{Z}$: the degree function of M defined by $\deg \xi := i$ if $0 \neq \xi \in M_i$.

(4) $l(M)$: the length of M, i.e., the largest length of chains of submodules of M.

(5) $\mathrm{depth}(M)$: the depth of M, i.e., the length of maximal regular sequence on M if $M \neq 0$ and $\mathrm{depth}(0) := \infty$.

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(6) $\mathrm{pd}_R(M)$: the projective dimension of M.

(7) $\mu(M) := l(M/\mathfrak{m}M)$: the number of minimal generators of M.

(8) $\mathrm{D}(M) := \{i \in \mathbb{Z} \mid (M/\mathfrak{m}M)_i \neq 0\}$: the set of degrees of minimal generators of M, especially $\mathrm{D}(0) = \varnothing$.

(9) $d(M) := \min\{i \in \mathbb{Z} \mid (M/\mathfrak{m}M)_i \neq 0\}$: the minimal degree of minimal generators of M, especially $d(0) = \infty$.

(10) $d'(M) := \max\{i \in \mathbb{Z} \mid (M/\mathfrak{m}M)_i \neq 0\}$: the maximal degree of minimal generators of M, especially $D(0) = -\infty$.

(11) $\mathrm{NZD}_1(M) := \{z \in R_1 \mid z \in \mathrm{NZD}(M)\}$ if $M \neq 0$ and $\mathrm{NZD}_1(0) := R_1 \setminus \{0\}$.

(12) $N \underset{M}{:} \mathfrak{m}^\infty := \bigcup_{i \in \mathbb{Z}} \left(N \underset{M}{:} \mathfrak{m}^i\right) = \bigcup_{i \geq 0} \left(N \underset{M}{:} \mathfrak{m}^i\right)$: the saturation of N *in* M.

(13) $\sigma(M) := 1 + \mathrm{top}\left(0 \underset{M}{:} \mathfrak{m}\right)$: 1+ the degree of the top socle of M, especially, $\sigma(M) := -\infty$ if $0 \underset{M}{:} \mathfrak{m} = 0$.

Definition 2.2. *Let* $N \subseteq M \in \mathcal{A}$.

(1) N is called $\mathfrak{m}$*-full in* M if $\mathfrak{m}N \underset{M}{:} z = N$ for some $0 \neq z \in R_1$. Especially in this case, we say that N is $\mathfrak{m}$*-full in* M *with respect to* z.

(2) $\mathfrak{m}\text{-full}(N;M) := \{z \mid N \text{ is } \mathfrak{m}\text{-full in } M \text{ w.r.t. } z \in R_1\}$ is called the *set of* $\mathfrak{m}$*-full divisors for* N *in* M.

Theorem 2.3. *(See (8, Theorem 3.6).) Let* $N \subseteq M \in \mathcal{A}$. *Then* $\mathfrak{m}\text{-full}(N;M)$ *is a Zariski open subset of* R_1.

Proposition 2.4. *(See (8, Proposition 3.4).)Let* $N \subseteq M \in \mathcal{A}$ *and* $z \in R_1$. *Then the following are equivalent:*

(i) N *is* $\mathfrak{m}$*-full in* M *w.r.t.* z;

(ii) $\mu(N) = l\left(0 \underset{M/N}{:} z\right) + \mu((N + zM)/zM)$.

Definition 2.5. *Let* $N \subseteq M \in \mathcal{A}$. $\delta_M(0) := 0$, $\delta_M(N) := \max\{0, \sigma(M/N) - d\}$ if $0 \neq N = N_{\langle d \rangle}$ and for general $N \neq 0$, we define $\delta_M(N) := \max\left\{\delta_M\left(N_{\langle i \rangle}\right) \middle| i \in \mathrm{D}(N)\right\}$. We call $\delta_M(N)$ the *componentwise* $\mathfrak{m}$*-full defect of* N *in* M if $\mathrm{depth} M \geq 1$.

Lemma 2.6. *Let* $0 \neq N \subseteq M \in \mathcal{A}$, $N_i := N_{\langle d_i \rangle}$ *and* $L_i := N_i \underset{M}{:} \mathfrak{m}^\infty$ $(i = 1, \cdots, r)$ *where* $\mathrm{D}(N) = \{d_1 < \cdots < d_r\}$. *Assume* $\mathrm{depth} M \geq 1$. *If* $j \geq \delta_M(N)$, then $\mathfrak{m}^j N = \mathfrak{m}^j N_1 + \cdots + \mathfrak{m}^j N_r = (L_1)_{\geq d_1 + j} + \cdots + (L_r)_{\geq d_r + j}$.

Proof. By [8, Lemma 4.9 (4)], $\delta_M(\mathfrak{m}^j N) = \max\{0, \delta_M(N) - j\} = 0$ since $j \geq \delta_M(N)$. Hence by definition, $\delta_M(\mathfrak{m}^j N_i) = 0$ $(i = 1, \cdots, r)$. By [8, Lemma 4.7 (1) (iv)], $\mathfrak{m}^j N_i = (L_i)_{\geq d_i + j}$ $(i = 1, \cdots, r)$. This completes the proof. □

3. Strongly $\mathfrak{m}$ - full modules

Definition 3.1. Let $N \subseteq M \in \mathcal{A}$. N is called *strongly* $\mathfrak{m}$*-full in* M if there exists $z \in R_1$ such that $\mathfrak{m}^i N \underset{M}{:} z^i = N$ for all $i \in \mathbb{Z}_{\geq 0}$. Especially in this case, we say that N is strongly $\mathfrak{m}$-full in M *with respect to* z.

Remark 3.2. *Let* $N \subseteq M \in \mathcal{A}$. If N is strongly $\mathfrak{m}$-full in M, then by definition, N is $\mathfrak{m}$-full in M. 0 is strongly $\mathfrak{m}$-full in M if and only if $\mathrm{NZD}_1(M) \neq \varnothing$.

Notation 3.3. Let $N \subseteq M \in \mathcal{A}$ and $z \in R_1$.

(1) $\widetilde{N}^{M} := \bigcup_{i \geq 0} \left(\mathfrak{m}^i N \underset{M}{:} \mathfrak{m}^i \right)$.

(2) $\widetilde{N}^{(M;z)} := \bigcup_{i \geq 0} \left(\mathfrak{m}^i N \underset{M}{:} z^i \right)$.

Remark 3.4. Let $N \subseteq M \in \mathcal{A}$. N is strongly $\mathfrak{m}$ -full in M w.r.t. $z \in R_1$ if and only if $\widetilde{N}^{(M;z)} = N$.

Lemma 3.5. *Let* $0 \neq L \subseteq M \in \mathcal{A}$ *. Assume that* $\operatorname{depth} M \geq 1$ *and* L *is a saturated graded submodule of* M *, i.e.,* $L \underset{M}{:} \mathfrak{m}^\infty = L$ *. Then* $\operatorname{NZD}_1(M/L) \cap \operatorname{NZD}_1(M) \neq \varnothing$ *and for any* $j \in \mathbb{Z}$ *,* $L_{\geq j}$ *is strongly* $\mathfrak{m}$ *-full w.r.t.* $z \in \operatorname{NZD}_1(M/L) \cap \operatorname{NZD}_1(M) \neq \varnothing$ *, i.e.,* $\mathfrak{m}^i L_{\geq j} \underset{M}{:} z^i = L_{\geq j}$ *for all* $i \in \mathbb{Z}_{\geq 0}$.

Proof. Since $\operatorname{depth} M \geq 1$, and $M/L = 0$ or $\operatorname{depth}(M/L) \geq 1$, we have $\operatorname{NZD}_1(M) \neq \varnothing$ and $\operatorname{NZD}_1(M/L) \neq \varnothing$. Moreover $\operatorname{NZD}_1(M)$ and $\operatorname{NZD}_1(M/L)$ are Zariski open subsets in R_1, so $\operatorname{NZD}_1(M/L) \cap \operatorname{NZD}_1(M) \neq \varnothing$. Obviously, $L_{\geq j} \subseteq \mathfrak{m}^i L_{\geq j} \underset{M}{:} z^i$. On the other hand, we remark that $\mathfrak{m}^i L_{\geq j} \underset{M}{:} z^i = \mathfrak{m}^i L_{\geq j} \underset{L}{:} z^i$ since $l\left(L / \mathfrak{m} L_{\geq j}\right) < \infty$, so $L = \mathfrak{m} L_{\geq j} \underset{M}{:} \mathfrak{m}^\infty$ and $z \in \operatorname{NZD}_1(M/L)$. Let $0 \neq \xi \in \mathfrak{m}^i L_{\geq j} \underset{L}{:} z^i$ be a homogeneous element, then $0 \neq z^i \xi \in \mathfrak{m}^i L_{\geq j} \subseteq L_{\geq j+i}$. Hence we have $\deg z^i \xi = \deg \xi + i \geq j + i$. This implies $\xi \in L_{\geq j}$, so $\mathfrak{m}^i L_{\geq j} \underset{M}{:} z^i = \mathfrak{m}^i L_{\geq j} \underset{L}{:} z^i \subseteq L_{\geq j}$. We are done. □

Proposition 3.6. *Let* $0 \neq L_1 \subseteq \cdots \subseteq L_p \subseteq M \in \mathcal{A}$ *be a chain of saturated graded submodules,* $e_1, \cdots, e_p \in \mathbb{Z}$ *with* $e_1 < \cdots < e_p$ *and* $N' := \left(L_1\right)_{\geq e_1} + \cdots + \left(L_p\right)_{\geq e_p}$ *. Assume* $\operatorname{depth} M \geq 1$ *. Then* N' *is strongly* $\mathfrak{m}$ *-full w.r.t.* $z \in \bigcap_{i=1}^{p} \operatorname{NZD}_1(M/L_i) \cap \operatorname{NZD}_1(M)$*, i.e.,* $\mathfrak{m}^i N' \underset{M}{:} z^i = N'$ *for all* $i \in \mathbb{Z}_{\geq 0}$.

Proof. The inclusion $N' \subseteq \mathfrak{m}^i N' \underset{M}{:} z^i$ obviously holds. On the other hand, let $0 \neq \xi \in \mathfrak{m}^i N' \underset{M}{:} z^i$ be a homogeneous element, then $0 \neq z^i \xi \in \mathfrak{m}^i N' = \mathfrak{m}^i \left(L_1\right)_{\geq e_1} + \cdots + \mathfrak{m}^i \left(L_p\right)_{\geq e_p}$. Let $c := \deg z^i \xi$, then $e_q + i \leq c < e_{q+1} + i$ for some $1 \leq q \leq p-1$ or $e_q + i \leq c$ with $q = p$, so we have:

$$0 \neq z^i \xi \in \left(\mathfrak{m}^i N'\right)_c = \left(\mathfrak{m}^i \left(L_1\right)_{\geq e_1} + \cdots + \mathfrak{m}^i \left(L_q\right)_{\geq e_q}\right)_c = \mathfrak{m}^i \left(L_1\right)_{c-i} + \cdots + \mathfrak{m}^i \left(L_q\right)_{c-i} = \mathfrak{m}^i \left(L_q\right)_{c-i} \subseteq \mathfrak{m}^i \left(L_q\right)_{\geq e_q} .$$

Hence we have $\xi \in \mathfrak{m}^i \left(L_q\right)_{\geq e_q} \underset{M}{:} z^i = \left(L_q\right)_{\geq e_q} \subseteq N'$ by Lemma 3.5. This implies $\mathfrak{m}^i N' \underset{M}{:} z^i \subseteq N'$ and completes the proof. □

Remark 3.7. In the proof of Proposition 3.6, we use the fact that $\left(\mathfrak{m}^i M\right)_{i+j} = \mathfrak{m}_1{}^i M_j + \mathfrak{m}_1{}^{i+1} M_{j-1} + \cdots = \mathfrak{m}_1{}^i M_j$ for any $i \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}$ and $M \in \mathcal{A}$.

Definition 3.8. Let $N \subseteq M \in \mathcal{A}$. $\widetilde{N}$ is called the *strongly* $\mathfrak{m}$ *-full closure of* N *in* M if it satisfies the following properties:

(1) $\widetilde{N}$ is a graded submodule of M , containing N and strongly $\mathfrak{m}$ -full in M .

(2) $\widetilde{N} \subseteq N'$, for any N' a strongly $\mathfrak{m}$ -full graded submodule of M , containing N .

That is to say, $\widetilde{N}$ is the minimal, with respect to inclusion relation, strongly $\mathfrak{m}$ -full graded submodule of M , containing N .

Remark 3.9. Let $0 \neq N \subseteq M \in \mathcal{A}$, $N_i := N_{\langle d_i \rangle}$ and $L_i := N_i \underset{M}{:} \mathfrak{m}^\infty$ $(i = 1, \cdots, r)$ *where* $\operatorname{D}(N) = \{d_1 < \cdots < d_r\}$. Then $L_1 \subseteq \cdots \subseteq L_r$ since $l\left(\left(N_1 + \cdots + N_i\right)/N_i\right) < \infty$, so $L_i = \left(N_1 + \cdots + N_i\right) \underset{M}{:} \mathfrak{m}^\infty$ for $i = 1, \cdots, r$.

Theorem 3.10. *Let* $0 \neq N \subseteq M \in \mathcal{A}$ *,* $N_i := N_{\langle d_i \rangle}$ *and* $L_i := N_i \underset{M}{:} \mathfrak{m}^\infty$ $(i = 1, \cdots, r)$ *where* $\mathrm{D}(N) = \{d_1 < \cdots < d_r\}$. *Assume* $\mathrm{depth} M \geq 1$ *. Then the following hold:*

(1) $\widetilde{N}^M = \widetilde{N}^{(M,z)} = (L_1)_{\geq d_1} + \cdots + (L_r)_{\geq d_r}$ *for any* $z \in \bigcap_{i=1}^{r} \mathrm{NZD}_1(M / L_i) \cap \mathrm{NZD}_1(M)$.

(2) $\widetilde{N}^M$ *is strongly* $\mathfrak{m}$ *-full.*

(3) *If* N *is strongly* $\mathfrak{m}$ *-full w.r.t. z, then* $N = \widetilde{N}^{(M,z)} = \widetilde{N}^M$ *. Especially,* $\widetilde{\left(\widetilde{N}^M\right)}^M = \widetilde{N}^M$.

(4) $\widetilde{N}^M$ *is the strongly* $\mathfrak{m}$ *-full closure of* N *in* M *.*

Proof. (1) By Lemma 2.6, if $j \geq \delta_M(N)$, then $\mathfrak{m}^j N = \mathfrak{m}^j N_1 + \cdots + \mathfrak{m}^j N_r = (L_1)_{\geq d_1 + j} + \cdots + (L_r)_{\geq d_r + j}$. Using this result, we have the following:

$$(L_1)_{\geq d_1} + \cdots + (L_r)_{\geq d_r} \subseteq \mathfrak{m}^j (L_1)_{\geq d_1} + \cdots + \mathfrak{m}^j (L_r)_{\geq d_r} \underset{M}{:} \mathfrak{m}^j \subseteq (L_1)_{\geq d_1 + j} + \cdots + (L_r)_{\geq d_r + j} \underset{M}{:} \mathfrak{m}^j = \mathfrak{m}^j N \underset{M}{:} \mathfrak{m}^j ,$$

$$\mathfrak{m}^j N \underset{M}{:} \mathfrak{m}^j = (L_1)_{\geq d_1 + j} + \cdots + (L_r)_{\geq d_r + j} \underset{M}{:} \mathfrak{m}^j \subseteq \mathfrak{m}^j N \underset{M}{:} z^j = (L_1)_{\geq d_1 + j} + \cdots + (L_r)_{\geq d_r + j} \underset{M}{:} z^j = (L_1)_{\geq d_1} + \cdots + (L_r)_{\geq d_r} .$$

This implies $\mathfrak{m}^j N \underset{M}{:} \mathfrak{m}^j = \mathfrak{m}^j N \underset{M}{:} z^j = (L_1)_{\geq d_1} + \cdots + (L_r)_{\geq d_r}$ for any $j \geq \delta_M(N)$, so we have:

$$\widetilde{N}^M = \widetilde{N}^{(M,z)} = (L_1)_{\geq d_1} + \cdots + (L_r)_{\geq d_r} .$$

(2) This follows from (1) and Proposition 3.6.

(3) From the definition of strong m-fullness, N is strongly $\mathfrak{m}$ -full in M if and only if $\widetilde{N}^{(M;z)} = N$. In general, $N \subseteq \widetilde{N}^M \subseteq \widetilde{N}^{(M;z)}$, hence $N = \widetilde{N}^M = \widetilde{N}^{(M;z)}$ if N is strongly $\mathfrak{m}$ -full in M . Especially, $\widetilde{N}^M$ is strongly $\mathfrak{m}$ -full by (2). Hence we have $\widetilde{\left(\widetilde{N}^M\right)}^M = \widetilde{N}^M$.

(4) $\widetilde{N}^M \supseteq N$ is strongly $\mathfrak{m}$ -full by (2). If $N' \supseteq N$ is a strongly $\mathfrak{m}$ -full graded submodule of M , then $N' = \widetilde{N'}^M \supseteq \widetilde{N}^M$ by (3). This implies that $\widetilde{N}^M$ is the strongly $\mathfrak{m}$ -full closure of N in M . □

Directly from Theorem 3.10, we get the following criteria for strong $\mathfrak{m}$ - fullness.

Corollary 3.11. *Let* $N \subseteq M \in \mathcal{A}$ *. Assume* $\mathrm{depth} M \geq 1$ *. Then the following are equivalent:*

(i) *N is strongly* $\mathfrak{m}$ *-full;*

(ii) $\widetilde{N}^M = N$ *;*

(iii) $N = 0$ *or*

$N = (L_1)_{\geq d_1} + \cdots + (L_r)_{\geq d_r}$ *, where* $N_i := N_{\langle d_i \rangle}$ *and* $L_i := N_i \underset{M}{:} \mathfrak{m}^\infty$ $(i = 1, \cdots, r)$ *with* $\mathrm{D}(N) = \{d_1 < \cdots < d_r\}$.

4. Componentwise $\mathfrak{m}$ - full modules and $\mathfrak{m}$ - adically $\mathfrak{m}$ - full modules

Definition 4.1. *Let* $N \subseteq M \in \mathcal{A}$ *.*

(1) N is called *componentwise* $\mathfrak{m}$ *-full* (*c.w.* $\mathfrak{m}$ *-full* for short) *in* M if $N_{\langle i \rangle}$ is $\mathfrak{m}$ -full in M for all $i \in \mathbb{Z}$.

(2) N is called $\mathfrak{m}$ - *adically* $\mathfrak{m}$ *-full* ($\mathfrak{m}$ *-ad.* $\mathfrak{m}$ *-full* for short) *in* M if $\mathfrak{m}^i N$ is $\mathfrak{m}$ -full in M for all $i \in \mathbb{Z}$.

Proposition 4.2. *(See (8, Proposition 4.2).) Let* $N \subseteq M \in \mathcal{A}$ *. If* N *is componentwise* $\mathfrak{m}$ *-full in* M *, then* N *is* $\mathfrak{m}$ *-full in* M *.*

Theorem 4.3. *(See (8, Theorem 4.10).) Let* $0 \neq N \subseteq M \in \mathcal{A}$ *and* $N_i := N_{\langle d_i \rangle}$ *,* $L_i := N_i \underset{M}{:} \mathfrak{m}^\infty$ $(i = 1, \cdots, r)$ *where* $\mathrm{D}(N) = \{d_1 < \cdots < d_r\}$ *. If* $\mathrm{depth} M \geq 1$ *, then the following are equivalent:*

(i) N *is componentwise* $\mathfrak{m}$ *-full in* M *;*

(ii) N_i *is* $\mathfrak{m}$*-full in* M *for all* $1 \le i \le r$ *;*
(iii) N_i *is* $\mathfrak{m}$*-full in* L_i *for all* $1 \le i \le r$ *;*
(iv) $N_i = \left(L_i\right)_{\ge d_i}$ *for all* $1 \le i \le r$ *;*
(v) $N = \left(L_1\right)_{\ge d_1} + \cdots + \left(L_r\right)_{\ge d_r}$ *and* $d_i = d'\left(L_i\right)$ *for all* $1 \le i \le r$ *.*

Corollary 4.4. *Let* $N \subseteq M \in \mathcal{A}$ *. Assume* $\operatorname{depth} M \ge 1$*. If* N *is componentwise* $\mathfrak{m}$*-full in* M *, then strongly* $\mathfrak{m}$*-full in* M *.*

Proof. If $N = 0$, there is nothing to prove. So we assume that $N \ne 0$. If N is componentwise $\mathfrak{m}$-full in M, then $N = \left(L_1\right)_{\ge d_1} + \cdots + \left(L_r\right)_{\ge d_r}$, where $N_i := N_{\langle d_i \rangle}$, $L_i := N_i \underset{M}{:} \mathfrak{m}^\infty$ $(i = 1, \cdots, r)$ with $\mathrm{D}(N) = \{d_1 < \cdots < d_r\}$ by Theorem 4.3 (iv). Hence N is strongly $\mathfrak{m}$-full in M by Corollary 3.11 (iii). □

Corollary 4.5. *Let* $N \subseteq M \in \mathcal{A}$ *. Assume* $\operatorname{depth} M \ge 1$*. If* N *is componentwise* $\mathfrak{m}$*-full in* M *, then* $\mathfrak{m}^j N$ *is also componentwise* $\mathfrak{m}$*-full in* M *for any* $j \in \mathbb{Z}_{\ge 0}$*.*

Proof. If $N = 0$, there is nothing to prove. So we assume that $N \ne 0$. Here we remark that if $N = N_1 + \cdots + N_r$, where $N_i := N_{\langle d_i \rangle}$ $(i = 1, \cdots, r)$ with $\mathrm{D}(N) = \{d_1 < \cdots < d_r\}$, then $\mathfrak{m}^j N = \mathfrak{m}^j N_1 + \cdots + \mathfrak{m}^j N_r$ for $j \in \mathbb{Z}_{\ge 0}$, so we have $\mathrm{D}\left(\mathfrak{m}^j N\right) \subseteq \{d_1 + j < \cdots < d_r + j\}$ and $\left\{\left(\mathfrak{m}^j N\right)_{\langle p \rangle} \middle| p \in \mathrm{D}(N)\right\} \subseteq \left\{\mathfrak{m}^j N_1, \cdots, \mathfrak{m}^j N_r\right\}$. On the other hand, by Theorem 4.3 (iii), $N_i = \left(L_i\right)_{\ge d_i}$ where $L_i := N_i \underset{M}{:} \mathfrak{m}^\infty = \mathfrak{m}^j N_i \underset{M}{:} \mathfrak{m}^\infty$ $(1 \le i \le r)$, so we have $\mathfrak{m}^j N_i = \mathfrak{m}^j \left(L_i\right)_{\ge d_i} = \left(L_i\right)_{\ge d_i + j}$ for $1 \le i \le r$. This implies $\left(\mathfrak{m}^j N\right)_{\langle p \rangle} = \left(\left(\mathfrak{m}^j N\right)_{\langle p \rangle} \underset{M}{:} \mathfrak{m}^\infty\right)_{\ge p}$ for all $p \in \mathrm{D}(N)$. Again by Theorem 4.3, $\mathfrak{m}^j N$ is componentwise $\mathfrak{m}$-full in M. □

Lemma 4.6. *Let* $0 \ne N \subseteq M \in \mathcal{A}$ *and* $N_i := N_{\langle d_i \rangle}$, $L_i := N_i \underset{M}{:} \mathfrak{m}^\infty$ $(i = 1, \cdots, r)$ *where* $\mathrm{D}(N) = \{d_1 < \cdots < d_r\}$*. If* $\operatorname{depth} M \ge 1$ *and* N *is* $\mathfrak{m}$*-adically* $\mathfrak{m}$*-full in* M *, then* $N = \left(L_1\right)_{\ge d_1} + \cdots + \left(L_r\right)_{\ge d_r}$ *and* $d_1 = d'\left(L_1\right)$*. Especially,* N *is strongly* $\mathfrak{m}$*-full.*

Proof. If N is $\mathfrak{m}$-adically $\mathfrak{m}$-full in M, then $\mathfrak{m}^j N \subseteq \mathfrak{m}^{j+1} N \underset{M}{:} \mathfrak{m} \subseteq \mathfrak{m}^{j+1} N \underset{M}{:} z_j = \mathfrak{m}^j N$ for each $j \in \mathbb{Z}_{\ge 0}$ and some $z_j \in R_1$. This implies $\mathfrak{m}^{j+1} N \underset{M}{:} \mathfrak{m} = \mathfrak{m}^j N$ for all $j \in \mathbb{Z}_{\ge 0}$. Hence we have:

$$\mathfrak{m}^{j+1} N \underset{M}{:} \mathfrak{m}^{j+1} = \left(\mathfrak{m}^{j+1} N \underset{M}{:} \mathfrak{m}\right) \underset{M}{:} \mathfrak{m}^j = \mathfrak{m}^j N \underset{M}{:} \mathfrak{m}^j = \cdots = N.$$

So we conclude that $N = \widetilde{N}^M = \left(L_1\right)_{\ge d_1} + \cdots + \left(L_r\right)_{\ge d_r}$, by Theorem 3.10 (1), especially N is strongly $\mathfrak{m}$-full by Corollary 3.11. Since $\mathfrak{m}^j N$ is also $\mathfrak{m}$-adically $\mathfrak{m}$-full, $d_1 + j \in \mathrm{D}\left(\mathfrak{m}^j N\right)$, $\left(\mathfrak{m}^j N\right)_{\langle d_1 + j \rangle} = \mathfrak{m}^j N_1$ and $L_1 = \mathfrak{m}^j N_1 \underset{M}{:} \mathfrak{m}^\infty$ for each $j \in \mathbb{Z}_{\ge 0}$, we have:

$$\mathfrak{m}^j N = \left(L_1\right)_{\ge d_1 + j} + \cdots = \mathfrak{m}^j \left(L_1\right)_{\ge d_1} + \cdots + \mathfrak{m}^j \left(L_r\right)_{\ge d_r}.$$

Comparing degree $d_1 + j$ part of the above equation, we see that $\left(L_1\right)_{d_1 + j} = \mathfrak{m}_1^j \left(L_1\right)_{d_1}$ for each $j \in \mathbb{Z}_{\ge 0}$. This implies $L_1 = \left(L_1\right)_{\langle d_1 \rangle}$, so $d_1 = d'\left(L_1\right)$. □

Lemma 4.7. *Let* $0 \ne N \subseteq M \in \mathcal{A}$ *and* $N_i := N_{\langle d_i \rangle}$ $(i = 1, \cdots, r)$*, where* $\mathrm{D}(N) = \{d_1 < \cdots < d_r\}$*. If* $\operatorname{depth} M \ge 1$*,* $r \ge 2$ *and* N *is* $\mathfrak{m}$*-adically* $\mathfrak{m}$*-full in* M *, then* $N' := N_2 + \cdots + N_r$ *is also* $\mathfrak{m}$*-adically* $\mathfrak{m}$*-full in* M *.*

Proof. Since N is $\mathfrak{m}$-adically $\mathfrak{m}$-full in M, we have $\mathfrak{m}\text{-full}\left(\mathfrak{m}^j N;M\right)\neq\varnothing$ for each $j\in\mathbb{Z}_{\geq 0}$. Let $z_j\in\mathfrak{m}\text{-full}\left(\mathfrak{m}^j N;M\right)\cap\mathrm{NZD}_1\left(M\right)$, then $\mathfrak{m}^{j+1}N\underset{M}{:}z_j=\mathfrak{m}^j N$. The inclusion $\mathfrak{m}^j N'\subseteq\mathfrak{m}^{j+1}N'\underset{M}{:}z_j$ clearly holds. On the other hand, let $0\neq\xi\in\mathfrak{m}^{j+1}N'\underset{M}{:}z_j\subseteq\mathfrak{m}^{j+1}N\underset{M}{:}z_j=\mathfrak{m}^j N$ be a nonzero homogeneous element. Then we have $0\neq z_j\xi\in\mathfrak{m}^{j+1}N'=\mathfrak{m}^{j+1}N_2+\cdots+\mathfrak{m}^{j+1}N_r$. Hence $\deg\xi\geq d_2+j$ and we have:

$$\xi\in\left(\mathfrak{m}^j N\right)_{\geq d_2+j}=\left(\mathfrak{m}^j N_1+\mathfrak{m}^j N_2+\cdots+\mathfrak{m}^j N_r\right)_{\geq d_2+i}=\mathfrak{m}^j N_2+\cdots+\mathfrak{m}^j N_r=\mathfrak{m}^j N'.$$

Hence we have $\mathfrak{m}^{j+1}N'\underset{M}{:}z_j\subseteq\mathfrak{m}^j N'$. This completes the proof. □

Remark 4.8. We remark that $\mathfrak{m}\text{-full}\left(\mathfrak{m}^j N;M\right)\cap\mathrm{NZD}_1\left(M\right)\neq\varnothing$ for each integer $j\in\mathbb{Z}_{\geq 0}$ by Theorem 2.3.

Theorem 4.9. *Let* $0\neq N\subseteq M\in\mathcal{A}$, $N_i:=N_{\langle d_i\rangle}$ *and* $L_i:=N_i\underset{M}{:}\mathfrak{m}^\infty$ $(i=1,\cdots,r)$ *where* $\mathrm{D}\left(N\right)=\left\{d_1<\cdots<d_r\right\}$. *Assume* $\operatorname{depth}M\geq 1$, *then the following are equivalent:*

(i) N *is* $\mathfrak{m}$*-adically* $\mathfrak{m}$*-full in* M.

(ii) $N=\left(L_1\right)_{\geq d_1}+\cdots+\left(L_r\right)_{\geq d_r}$ *and* $d_i=d'\left(L_i\right)$ *for all* $1\leq i\leq r$.

(iii) N *is componentwise* $\mathfrak{m}$*-full in* M.

Proof. $(\mathrm{i})\Rightarrow(\mathrm{ii})$: By Lemma 4.7, $N=N_1+\cdots+N_r$, $N^{(2)}:=N_2+\cdots+N_r$, $N^{(3)}:=N_3+\cdots+N_r,\cdots,N^{(r)}:=N_r$ are all $\mathfrak{m}$-adically $\mathfrak{m}$-full in M. Therefore by Lemma 4.6, we have: $N=\left(L_1\right)_{\geq d_1}+\cdots+\left(L_r\right)_{\geq d_r}$ with $d_1=d'\left(L_1\right)$, $N^{(2)}=\left(L_2\right)_{\geq d_2}+\cdots+\left(L_r\right)_{\geq d_r}$ with $d_2=d'\left(L_2\right)$, $N^{(3)}=\left(L_3\right)_{\geq d_3}+\cdots+\left(L_r\right)_{\geq d_r}$ with $d_3=d'\left(L_3\right),\cdots,N^{(r)}=\left(L_r\right)_{\geq d_r}$ with $d_r=d'\left(L_r\right)$. This implies $N=\left(L_1\right)_{\geq d_1}+\cdots+\left(L_r\right)_{\geq d_r}$ and $d_i=d'\left(L_i\right)$ for all $1\leq i\leq r$.

$(\mathrm{ii})\Leftrightarrow(\mathrm{iii})$: This follows from Theorem 4.3.

$(\mathrm{iii})\Rightarrow(\mathrm{i})$: This follows from Proposition 4.2. and Corollary 4.5. □

Remark 4.10. If N is $\mathfrak{m}$-adically $\mathfrak{m}$-full in M and $\operatorname{depth}M\geq 1$, then from Theorem 4.9 and Theorem 3.10 (1),

$$\varnothing\neq\mathrm{NZD}_1\left(M\right)\cap\bigcap_{i=1}^{r}\mathrm{NZD}_1\left(M/L_i\right)\subseteq\mathrm{NZD}_1\left(M\right)\cap\bigcap_{i=1}^{r}\mathrm{NZD}_1\left(M/N_i\right)_{\geq d_i}\subseteq\bigcap_{j\in\mathbb{Z}_{\geq 0}}\mathfrak{m}\text{-full}\left(\mathfrak{m}^j N;M\right).$$

5. Application to ideals of the polynomial ring in two variables

Definition 5.1. *Let* $M\in\mathcal{A}$. $\operatorname{reg}\left(M\right):=\sup\left\{j-i\,\middle|\,i\in\mathbb{Z}_{\geq 0},j\in\mathbb{Z}\text{ with }\operatorname{Tor}_i^R\left(M,k\right)_j\neq 0\right\}$ is called the regularity of M.

Definition 5.2. *Let* $I\subseteq R=K[X_1,\cdots,X_r]$ be a proper (i.e. $I\neq R$) homogeneous ideal. I is called a componetwise linear ideal if $\operatorname{reg}\left(I_{\langle j\rangle}\right)=j$ for all $j\in\mathbb{Z}$ with $I_{\langle j\rangle}\neq 0$.

Definition 5.3 (Watanabe)**.** (See (11).) Let $I\subseteq R=K[X_1,\cdots,X_r]$ be a proper homogeneous ideal. I is called a completely $\mathfrak{m}$-full ideal if there exist $z_1,\cdots,z_n\in R_1$ such that I is $\mathfrak{m}$-full in R w.r.t. z_1 and $\left(I+z_1R+\cdots+z_iR\right)/\left(z_1R+\cdots+z_iR\right)$ is $\mathfrak{m}$-full in $R/\left(z_1R+\cdots+z_iR\right)$ w.r.t. z_{i+1} for each $i=1,\cdots,n-1$.

Remark 5.4. Let $I\subseteq R=K[X]$ be a proper homogeneous ideal of the polynomial ring in one variable. Then $I=0$ or $I=\left(X^i\right)$ for some integer $i\geq 1$ since R is a principal ideal domain. So I is $\mathfrak{m}$-full in R w.r.t. X.

We need the following result to prove Proposition 5.7.

Theorem 5.5 *(Harima and Watanabe)*. *(See (5, Theorem 1.1).)Let $I \subseteq R = K[X_1, \cdots, X_r]$ be an homogeneous ideal. Then I is a completely $\mathfrak{m}$ -full ideal if and only if I is a componetwise linear ideal.*

Remark 5.6. Recall we assume that K is an infinite field throughout this paper. If K is a finite field, then Theorem 5.5 does not hold.

From now on, we assume that $R = K[X, Y]$ is the polynomial ring in two variables. For any two elements $f, g \in R$, we denote $f|g$ if f divides g and $f \nmid g$ if f does not divide g.

Proposition 5.7. *Let $I \subseteq R = K[X, Y]$ be a proper homogeneous ideal. Then the following are equivalent:*

(i) *I is componetwise $\mathfrak{m}$ -full in R.*
(ii) *I is strongly $\mathfrak{m}$ -full in R.*
(iii) *I is $\mathfrak{m}$ -full in R.*
(iv) *I is a completely $\mathfrak{m}$ -full ideal.*
(v) *I is a componetwise linear ideal.*

Proof. $(\mathrm{i}) \Rightarrow (\mathrm{ii})$: This follows from Corollary 4.4. $(\mathrm{ii}) \Rightarrow (\mathrm{iii})$:This is clear by definition. $(\mathrm{iii}) \Rightarrow (\mathrm{iv})$: Since I is $\mathfrak{m}$ -full in R, there exists $0 \neq z \in R_1$ such that $\mathfrak{m}I \underset{R}{:} z = I$. Then $(I + zR)/(zR)$ is $\mathfrak{m}$ -full in $R / zR \simeq K[X]$ by Remark 5.4. Hence I is a completely $\mathfrak{m}$ -full ideal. $(\mathrm{iv}) \Leftrightarrow (\mathrm{v})$: This follows from Theorem 5.5. $(\mathrm{v}) \Rightarrow (\mathrm{i})$: Since I is a componetwise linear ideal, every component $I_{\langle j \rangle}$ ($j \in \mathbb{Z}$) is also a componetwise linear ideal by definition. Hence from Theorem 5.5, $I_{\langle j \rangle}$ is a completely $\mathfrak{m}$ -full , so especially $\mathfrak{m}$ -full ideal for each $j \in \mathbb{Z}$. This implies I is componetwise $\mathfrak{m}$ -full in R. □

Lemma 5.8. *Let $0 \neq J \subseteq R = K[X, Y]$ be a saturated homogeneous ideal, i.e., $J \underset{R}{:} \mathfrak{m}^{\infty} = J$, then J is a free module of rank 1, especially, is a principal ideal , i.e., $J = Rf$ for some homogeneous element $f \in R$.*

Proof. If $J = R$, there is nothing to prove. So we assume that $J \neq R$. Since $\operatorname{depth} R / J = 1$, by Auslander-Buchsbaum formula, we have the $\mathrm{pd}_R R / J = 1$, so $\mathrm{pd}_R J = 0$. By the graded Nakayama's lemma, J is a free module of rank 1. □

The next result, essentially appears in (11, Theorem 4). We have slightly extended this result as follows:

Lemma 5.9. *Let $I \subseteq R = K[X, Y]$ be a proper homogeneous ideal, $d := d(I)$, $J := I \underset{R}{:} \mathfrak{m}^{\infty}$, $J_1 := I_{\langle d \rangle} \underset{R}{:} \mathfrak{m}^{\infty}$ and $z \in \mathrm{NZD}_1(R / J_1) \cap \mathrm{NZD}_1(R / J) (\neq \varnothing)$. Then the following hold:*

(1) *There exist $0 \neq f, f_1 \in R$ such that $J = Rf$ and $J_1 = Rf_1$.*
(2) *There exists $0 \neq g \in I_d$ a degree d element in I such that $z \nmid g$.*
(3) $J \big/ (I + Jz) \simeq \left(K[X] \big/ (X^{d-e}) \right)(-e)$, *where $e := \deg f$.*
(4) $l\left(0 \underset{R/I}{:} z\right) = l\left(0 \underset{J/I}{:} z\right) = d - e$, *where $e := \deg f$.*

Proof. (1) This follows from Lemma 5.8 since J_1 and J are saturated homogeneous ideals.
(2) Assume $z|g$, for all $0 \neq g \in I_d$. Then $I_{\langle d \rangle} \subseteq Rz$ and we have:

$$Rf_1 = J_1 = I_{\langle d \rangle} \underset{R}{:} \mathfrak{m}^{\infty} \subseteq Rz \underset{R}{:} \mathfrak{m}^{\infty} = Rz.$$

Hence $z|f_1$, so $z \notin \mathrm{NZD}_1(R / J_1)$. This is a contradiction.
(3) Since $I \subseteq J = Rf$, there exists a homogenesou ideal I' such that $I = I'f$. Then we have:

$$J/(I+Jz) = Rf/(I'+Rz)f \simeq \left(R/(I'+Rz)\right)(-e).$$

On the other hand, there exists an isomorphism $\varphi : R/Rz \xrightarrow{\sim} K[X]$. By (2), there exists $0 \neq g \in I_d$ such that $z \nmid g$, so we have $(I'+Rz)/Rz = (Rg'+Rz)/Rz$, where $g = g'f$, and

$$\varphi((I'+Rz)/Rz) = \varphi((Rg'+Rz)/Rz) = (X^{d-e}) \subseteq K[X].$$

Hence $R/(I'+Rz) \simeq K[X]/(X^{d-e})$. This complete the proof.

(4) Since $z \in \mathrm{NZD}_1(R/J)$, we have $l(0 \underset{R/I}{:} z) = l(0 \underset{J/I}{:} z)$. From the following exact sequence:

$$0 \to (0 \underset{J/I}{:} z)(-1) \to (J/I)(-1) \xrightarrow{\times z} J/I \to J/(I+Jz) \to 0$$

we see that $l(0 \underset{J/I}{:} z) = l(J/(I+Jz)) = l(K[X]/(X^{d-e})) = d-e$ by (3) since $l(J/I) < \infty$. □

Lemma 5.10. *Let* $0 \neq I \subseteq R = K[X,Y]$ *be a proper homogeneous ideal,* $I_i := I_{\langle d_i \rangle}$ *and* $J_i := I_i \underset{R}{:} \mathfrak{m}^\infty$ $(i = 1, \cdots, r)$, *where* $\mathrm{D}(I) = \{d_1 < \cdots < d_r\}$. *Then the following hold:*

(1) *There exist homogeneous elements* $f_1, \cdots, f_r$ *in* R *with* $e_1 := \deg f_1 > \cdots > e_r := \deg f_r$ *such that* $J_i := Rf_i$ $(i = 1, \cdots, r)$ *and* $f_{i+1} | f_i$ $(i = 1, \cdots, r-1)$.

(2) *There exist ideals* $I'_1, \cdots, I'_r \subseteq R$ *such that* $I_1 = I'_1 f_1, \cdots, I_r = I'_r f_r$ *with* $d(I'_i) = d'(I'_i) = d_i - e_i$ $(i = 1, \cdots, r)$.

(3) $\tilde{I}^R := \mathfrak{m}^{d_1 - e_1} f_1 + \cdots + \mathfrak{m}^{d_r - e_r} f_r$.

(4) *If* $\mu(I) = d(I) + 1 - d(I \underset{R}{:} \mathfrak{m}^\infty)$, *then* $I = \tilde{I}^R$.

Proof. (1) follows form Lemma 5.8. and $J_1 \subseteq \cdots \subseteq J_r$ by Remark 3.9. (2) follows form (1). (3) follows form Theorem 3.10 (1) . (4) follows from Lemma 5.9 (4) and Proposition 2.4 since for $z \in \mathrm{NZD}_1(R/J_1) \cap \mathrm{NZD}_1(R/J)$, we have $l(0 \underset{R/I}{:} z) = d(I) - d(I \underset{R}{:} \mathfrak{m}^\infty)$ and $\mu((I+zR)/zR) = 1$. □

From Proposition 5.7 and Lemma 5.10, we have the following theorem:

Theorem 5.11. *Let* $I \subseteq R = K[X,Y]$ *be a proper homogeneous ideal. Then the following are equivalent:*

(i) *I is a componetwise linear ideal;*

(ii) I *is componetwise* $\mathfrak{m}$*-full in* R*;*

(iii) I *is strongly* $\mathfrak{m}$*-full in* R*;*

(iv) $I = \mathfrak{m}^{d_1 - c_1} f_1 + \cdots + \mathfrak{m}^{d_r - c_r} f_r$, *where* $f_1, \cdots, f_r \in R$ *homogeneous elements with* $c_1 := \deg f_1 > \cdots > c_r := \deg f_r$, $f_{i+1} | f_i$ $(i = 1, \cdots, r-1)$ *and* $\mathrm{D}(I) = \{d_1 < \cdots < d_r\}$*;*

(v) $\mu(I) = d(I) + 1 - d(I \underset{R}{:} \mathfrak{m}^\infty)$.

Remark 5.12. Theorem 5.10 (v) is an extension of the result (11, Theorem 4).

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