Paper

Componentwise $\mathfrak{m}$-full modules

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Abstract We introduce the componentwise $\mathfrak{m}$ - full property for a pair of graded modules, which is stronger than the $\mathfrak{m}$ - full property, and give criteria for checking whether a pair of graded modules has the componentwise $\mathfrak{m}$ - full property or not.

Keywords : Standard graded commutative algebra, Componentwise $\mathfrak{m}$ - full modules, Defect of componentwise $\mathfrak{m}$ - fullness, $\mathfrak{m}$ - full modules.

1. Intoroduction

The property of homogeneous ideals of a standard graded Noetherian commutative algebra over a field, called the $\mathfrak{m}$ -fullness, and related topics have been studied by many authors (e.g., (1)-(12)). The property, $\mathfrak{m}$ -fullness, can naturally be extended to the property of graded modules and their graded submodules.

In this paper, we introduce the componentwise $\mathfrak{m}$ -full property, which is stronger than the $\mathfrak{m}$ -full property, for a pair of a graded module and its graded submodule. We give criteria for componentwise $\mathfrak{m}$ -fullness. Especially, the defect of componentwise $\mathfrak{m}$ -fullness, which measures the $\mathfrak{m}$ -fullness, is introduced in Section 4.

In Section 2, the preliminary section, we fix some notations which we use here. Also we mention Zariski openness of the set $\Lambda(M)$ (see Notation 2.6), which plays an important role in in Section 3, in Lemma 2.7.

In Section 3, we study $\mathfrak{m}$ -fullness. The criteria for $\mathfrak{m}$ -fullness is given in Proposition 3.4. We also see that the locus of $\mathfrak{m}$ -full divisors (see Definition 3.1 (2)) is Zariski open (see Theorem 3.6).

Finally, in Section 4, we define componentwise $\mathfrak{m}$ -full property and the defect of componentwise $\mathfrak{m}$ -fullness, and give criteria for componentwise $\mathfrak{m}$ -fullness in Theorem 4.10.

2. Preliminaries

Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}_{\geq 0}$ be the set of nonnegative integers and R be a standard graded Noetherian commutative algebra over an infinite field K with the maximal homogeneous ideal $\mathfrak{m}$ and the residue field $k \simeq R/\mathfrak{m}$. Let $\mathcal{A}$ be the category of finitely generated graded R -modules, for $M \in \mathcal{A}$ and $i \in \mathbb{Z}$, we denote M_i the K -vector subspace generated by homogeneous elements of degree i in M, $M(i)$ the i -shifted module of M defined by $(M(i))_j := i + j$ for all $j \in \mathbb{Z}$. For a graded submodule N of M and a homogeneous ideal I of R ,we denote $N \underset{M}{:} I := \{\xi \in M \mid I\xi \subseteq N\}$, and similarly, for a homogeneous element $z \in R$, we also denote $N \underset{M}{:} z := \{\xi \in M \mid z\xi \subseteq N\}$. We fix some notations as follows:

Notation 2.1. Let $N \subseteq M \in \mathcal{A}$ and $j \in \mathbb{Z}$.

(1) $M_{\geq j} := \bigoplus_{i \geq j} M_i$: the graded submodule of elements of degrees greater than or equal to j in M.

(2) $M_{\langle j \rangle} := RM_j$: the graded submodule generated by elements of degrees j in M.

(3) $\mathfrak{m}^j := R$ and $z^j := 1$ if $j \leq 0$.

(4) $\deg : \coprod_{i \in \mathbb{Z}} (M_i \setminus \{0\}) \to \mathbb{Z}$: the degree function of M defined by $\deg \xi := i$ if $0 \neq \xi \in M_i$.

(5) $\operatorname{nzc}(M) := \{i \in \mathbb{Z} \mid M_i \neq 0\}$: the set of degrees of nonzero homogeneous components of M.

(6) $\operatorname{low}(M) := \inf\{i \in \mathbb{Z} \mid M_i \neq 0\}$ if $M \neq 0$ and $\operatorname{low}(0) := \infty$.

(7) $\operatorname{top}(M) := \sup\{i \in \mathbb{Z} \mid M_i \neq 0\}$ if $M \neq 0$ and $\operatorname{top}(0) := -\infty$.

(8) $\sigma(M) := 1 + \operatorname{top}(0 \underset{M}{:} \mathfrak{m})$: 1+ the degree of the top socle of M, especially, $\sigma(M) := -\infty$ if $0 \underset{M}{:} \mathfrak{m} = 0$.

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(9) $l(M)$: the length of M, i.e., the largest length of chains of submodules of M.
(10) $\operatorname{depth}(M)$: the depth of M, i.e., the length of maximal regular sequence on M if $M \neq 0$ and $\operatorname{depth}(0) := \infty$.
(11) $\mu(M) := l(M/\mathfrak{m}M)$: the number of minimal generators of M.
(12) $\mathrm{D}(M) := \operatorname{nzc}(M/\mathfrak{m}M)$: the set of degrees of minimal generators of M, especially $\mathrm{D}(0) = \varnothing$.
(13) $d(M) := \operatorname{low}(M/\mathfrak{m}M)$: the minimal degree of minimal generators of M, especially $d(0) = \infty$.
(14) $d'(M) := \operatorname{top}(M/\mathfrak{m}M)$: the maximal degree of minimal generators of M, especially $D(0) = -\infty$.
(15) $\mathrm{NZD}_{\geq 1}(M) := \left\{ z \in \mathfrak{m} \,\middle|\, 0 \underset{M}{:} z = 0 \right\}$ if $M \neq 0$: the set of nonzero divisors of M in $\mathfrak{m}$ and $\mathrm{NZD}_{\geq 1}(0) := \mathfrak{m} \setminus \{0\}$.
(16) $\mathrm{NZD}_1(M) := \left\{ z \in R_1 \,\middle|\, z \in \mathrm{NZD}(M) \right\}$ if $M \neq 0$ and $\mathrm{NZD}_1(0) := R_1 \setminus \{0\}$.
(17) $\operatorname{Ass}M$: the set of associated prime ideals of M.
(18) $N \underset{M}{:} \mathfrak{m}^\infty := \bigcup_{i \in \mathbb{Z}} \left(N \underset{M}{:} \mathfrak{m}^i \right) = \bigcup_{i \geq 0} \left(N \underset{M}{:} \mathfrak{m}^i \right)$: the saturation of N *in* M.

Remark 2.2. If $M = 0$ then $M_i = 0$ for all $i \in \mathbb{Z}$, $\mathrm{D}(0) = \operatorname{nzc}(0) = \varnothing$, $\operatorname{low}(0) = d(0) = \infty$, $\operatorname{top}(0) = d'(0) = \sigma(0) = -\infty$ and $\deg(0)$ is not defined. We assume $\mathbb{Z} \cup \{\infty\} \cup \{-\infty\}$ an ordered set with $\max(\mathbb{Z} \cup \{\infty\} \cup \{-\infty\}) = \infty$ and $\min(\mathbb{Z} \cup \{\infty\} \cup \{-\infty\}) = -\infty$. We also assume that $\infty + i =: \infty$ and $-\infty + i =: -\infty$ for all $i \in \mathbb{Z}$.

Remark 2.3. Let $N \subseteq M \in \mathcal{A}$, $\sigma := \sigma(M/N)$ and $L := N \underset{M}{:} \mathfrak{m}^\infty$. We remark the following:
(1) If $i \geq \sigma$, then $L_i = N_i$ since $0 \underset{M/N}{:} \mathfrak{m} \subseteq 0 \underset{M/N}{:} \mathfrak{m}^\infty = L/N$.
(2) $\sigma(M/N) = 1 + \operatorname{top}\left(0 \underset{M/N}{:} \mathfrak{m}\right) = 1 + \operatorname{top}\left(0 \underset{L/N}{:} \mathfrak{m}\right) = \sigma(L/N) = 1 + \operatorname{top}(L/N)$.
(3) If $\mathrm{NZD}_1(M) \neq \varnothing$ and $M_i \neq 0$, then $M_j \neq 0$ for any $j \geq i$.
(4) If $\mathrm{NZD}_1(M) \neq \varnothing$ and $0 \neq N \neq L$, then $d := d(N) \leq \sigma$. Actuary $L_{\sigma-1} \neq N_{\sigma-1}$ since $N \neq L$. This implies $L_{\sigma-1} \neq 0$ and $N_i = L_i \neq 0$ for all $i \geq \sigma$ by (1) and (2), especially $N_\sigma \neq 0$. Hence $d \leq \sigma$. The condition $\mathrm{NZD}_1(M) \neq \varnothing$ is crucial. In fact, if $N := k(-a) \subseteq M := N \oplus k(-b)$ and $a > b+1$, then $d(N) = a > \sigma(M/N) = b+1$.

In general, $\mathrm{NZD}_1(M)$ is a Zariski open subset of R_1 since $\mathrm{NZD}_1(M) = R_1 \setminus \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} (R_1 \cap \mathfrak{p})$ is a complement of finite union of K-linear subspaces $\mathfrak{m}_1 \cap \mathfrak{p}$ $(\mathfrak{p} \in \operatorname{Ass}M)$. The following Lemma gives a criterion for checking whether $\mathrm{NZD}_1(M)$ is empty or not.

Lemma 2.4. *Let* $M \in \mathcal{A}$. *Then the following hold:*
(1) $\operatorname{depth}(M) \geq 1$ *if and only if* $0 \underset{M}{:} \mathfrak{m} = 0$.
(2) $\mathrm{NZD}_1(M) \neq \varnothing$ *if and only if* $\operatorname{depth}(M) \geq 1$.

Proof. (1) First we remark that $\operatorname{depth}(0) = \infty$, so $M \neq 0$ if $\operatorname{depth}(M) = 0$. Since $\mathrm{NZD}_{\geq 1}(M) = \mathfrak{m} \setminus \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$, using prime avoidance, $\operatorname{depth}(M) < 1$, i.e., $\operatorname{depth}(M) = 0$ if and only if $\mathfrak{m} \in \operatorname{Ass}M$, if and only if $0 \underset{M}{:} \mathfrak{m} \neq 0$. Hence we are done.
(2) We remark that $\mathrm{NZD}_1(0) := R_1 \setminus \{0\}$ and $\operatorname{depth}(0) = \infty$. Hence if $\mathrm{NZD}_1(M) \neq \varnothing$, then clearly $\operatorname{depth}(M) \geq 1$. On the other hand, if $\operatorname{depth}(M) \geq 1$, then from (1), $\mathfrak{m} \notin \operatorname{Ass}M$, so $R_1 = \mathfrak{m}_1 \neq \mathfrak{m}_1 \cap \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}M$. Since $\operatorname{Ass}M$ is a finite set and each $\mathfrak{m}_1 \cap \mathfrak{p}$ $(\mathfrak{p} \in \operatorname{Ass}M)$ is a proper vector subspace of R_1 over an infinite field K, we

have $\mathrm{NZD}_1(M) = R_1 \setminus \bigcup_{\mathfrak{p}\in\mathrm{Ass}(M)} (\mathfrak{m}_1 \cap \mathfrak{p}) \neq \emptyset$. □

Remark 2.5. Lemma 2.4 says that those three conditions $\mathrm{depth}(M) \geq 1$, $0 \underset{M}{:} \mathfrak{m} = 0$ and $\mathrm{NZD}_1(M) \neq \emptyset$ are equivalent. Lemma 2.4. (1) holds without any assumption on the quotient field K, but the field K being infinite is necessary for Lemma 2.4 (2) to hold .

Notation 2.6. *Let* $M \in \mathcal{A}$.

(1) $\lambda(M) := \min\left\{ l\left(0 \underset{M}{:} \mathrm{z}\right) \,\middle|\, \mathrm{z} \in R_1 \right\}$.

(2) $\Lambda(M) := \left\{ \mathrm{z} \in R_1 \,\middle|\, l\left(0 \underset{M}{:} \mathrm{z}\right) = \lambda(M) \right\}$.

Lemma 2.7. *Let* $M \in \mathcal{A}$ *and* $L := 0 \underset{M}{:} \mathfrak{m}^\infty$. *Then the following hold:*

(1) $\lambda(M) < \infty$.

(2) $\mathrm{NZD}_1(M/L) \neq \emptyset$ and, then $0 \underset{M}{:} z = 0 \underset{L}{:} z$ if $z \in \mathrm{NZD}_1(M/L)$.

(3) *If* $z \in \Lambda(M)$, *then* $z \in \mathrm{NZD}_1(M/L)$.

(4) $\Lambda(L)$ *is a nonempty Zariski open subset of* R_1.

(5) $\lambda(M) = \lambda(L)$.

(6) $\Lambda(M) = \Lambda(L) \cap \mathrm{NZD}_1(M/L)$ *is a nonempty Zariski open subset of* R_1.

Proof. (1) From Lemma 2.4, $\mathrm{NZD}_1\left(M_{\geq\sigma(M)}\right) \neq \emptyset$ since $\left(0 \underset{M}{:} \mathfrak{m}\right)_{\geq\sigma(M)} = 0 \underset{M_{\geq\sigma(M)}}{:} \mathfrak{m} = 0$. If $\mathrm{z} \in \mathrm{NZD}_1\left(M_{\geq\sigma(M)}\right)$, then $0 \underset{M}{:} z \subseteq \bigoplus_{i<\sigma(M)} M_i$, hence $\lambda(M) \leq l\left(0 \underset{M}{:} z\right) \leq \sum_{i<\sigma(M)} \dim_K M_i < \infty$.

(2) Applying by Lemma 2.4, $\mathrm{NZD}_1(M/L) \neq \emptyset$ since $0 \underset{M/L}{:} \mathfrak{m} = 0$. If $z \in \mathrm{NZD}_1(M/L)$, then $0 \underset{M}{:} z \subseteq L$, hence $0 \underset{M}{:} z = 0 \underset{L}{:} z$.

(3) If $z \in \Lambda(M)$, then $l\left(0 \underset{M}{:} z\right) = \lambda(M) < \infty$ by (1), hence $0 \underset{M}{:} z \subseteq L$. This implies $z \in \mathrm{NZD}_1(M/L)$.

(4) The condition that $\mathrm{z} \in R_1$ belongs to $\Lambda(L)$ gives a maximal rank condition for the linear map $\times \mathrm{z}$ on the finite dimensional K-vector space L, since $l(L) < \infty$. Hence $\Lambda(L)$ is a nonempty Zariski open subset of R_1 .

(5) Since both $\Lambda(L)$ and $\cap \mathrm{NZD}_1(M/L)$ are nonempty Zariski open subsets of R_1 by (2) and (4), we can chose an element $z \in \Lambda(L) \cap \mathrm{NZD}_1(M/L) \neq \emptyset$ and $\lambda(L) = l\left(0 \underset{L}{:} \mathrm{z}\right) = l\left(0 \underset{M}{:} \mathrm{z}\right) \leq \lambda(M)$. On the other hand, if $z \in \Lambda(M)$, then $\lambda(M) = l\left(0 \underset{M}{:} \mathrm{z}\right) = l\left(0 \underset{L}{:} \mathrm{z}\right) \geq \lambda(L)$ by (3). Hence $\lambda(M) = \lambda(L)$.

(6) If $z \in \Lambda(M)$, then $l\left(0 \underset{M}{:} \mathrm{z}\right) = l\left(0 \underset{L}{:} \mathrm{z}\right) = \lambda(M) = \lambda(L)$, i.e., $z \in \Lambda(L)$ by (5) and $z \in \mathrm{NZD}_1(M/L)$ by (3). Hence we have $\Lambda(M) \subseteq \Lambda(L) \cap \mathrm{NZD}_1(M/L)$. On the other hand, if $z \in \Lambda(L) \cap \mathrm{NZD}_1(M/L)$, then $l\left(0 \underset{M}{:} \mathrm{z}\right) = l\left(0 \underset{L}{:} \mathrm{z}\right) = \lambda(L) = \lambda(M)$ by (2), (3) and (5). This implies $\Lambda(L) \cap \mathrm{NZD}_1(M/L) \subseteq \Lambda(M)$, we are done. □

3. $\mathfrak{m}$-fullness

Definition 3.1. *Let* $N \subseteq M \in \mathcal{A}$.

(1) N is called $\mathfrak{m}$*-full in* M if $\mathfrak{m}N \underset{M}{:} z = N$ for some $0 \neq z \in R_1$. Especially in this case, we say that N is $\mathfrak{m}$*-full in* M *with respect to* z.

(2) $\mathfrak{m}\text{-full}(N;M) := \{z \mid N \text{ is } \mathfrak{m}\text{-full in } M \text{ w.r.t. } z \in R_1\}$ is called the *set of* $\mathfrak{m}$*-full divisors for* N *in* M.

Remark 3.2. If $0 \neq z \in R_1$, then $\mathfrak{m}M \underset{M}{:} z = M$. Hence M is $\mathfrak{m}$-full in M. By definition, 0 is $\mathfrak{m}$-full in M w.r.t. $z \in R_1 \setminus \{0\}$ if and only if $z \in \mathrm{NZD}_1(M)$.

We use the following Lemma 3.3 to prove Proposition 3.4.

Lemma 3.3. *Given a commutative diagram in* $\mathcal{A}$ *with exact lows*

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & f_A\downarrow & & f_B\downarrow & \swarrow_g & \downarrow f_C & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array},$$

and let the part of the ker-coker sequence induced by the above exact sequence be

$$\mathrm{Ker}(f_B) \xrightarrow{\alpha} \mathrm{Ker}(f_C) \xrightarrow{\beta} \mathrm{Coker}(f_A).$$

Then we have $\mathrm{Im}(\alpha) = \mathrm{Ker}(g) = \mathrm{Ker}(\beta)$.

Proof. It is easy to check that $\mathrm{Im}(\alpha) \subseteq \mathrm{Ker}(g) \subseteq \mathrm{Ker}(\beta)$ by diagram chasing. Hence $\mathrm{Im}(\alpha) = \mathrm{Ker}(g) = \mathrm{Ker}(\beta)$. □

In the case of ideals, the next Proposition 3.4, essentially appears in (5, Lemma 4.3). But we state the proof for the convenience of the readers.

Proposition 3.4. *Let* $N \subseteq M \in \mathcal{A}$ *and* $z \in R_1$. *Then the following are equivalent:*

(i) N *is* $\mathfrak{m}$*-full in* M *w.r.t.* z *;*

(ii) $\varphi^{(z)} : M/\mathfrak{m}N \to (M/N)(1)$ *is injective, where* $\varphi^{(z)}$ *is a map defined by multiplication by* z *;*

(iii) $\mu(N) = l\left(0 \underset{M/\mathfrak{m}N}{:} z\right)$*;*

(iv) $N/\mathfrak{m}N \simeq 0 \underset{M/\mathfrak{m}N}{:} z$ *;*

(v) $\mu(N) = l\left(0 \underset{M/N}{:} z\right) + \mu\left((N+zM)/zM\right)$*;*

(vi) $N/\mathfrak{m}N \simeq \left(0 \underset{M/N}{:} z\right)(-1) \oplus (N+zM)/(\mathfrak{m}N+zM)$.

Proof. (i) is equivalent to (iv) from the definition of $\mathfrak{m}$-fullness. Applying Lemma 3.3 to the following commutative diagram with exact lows:

$$\begin{array}{ccccccccc} 0 & \to & (N/\mathfrak{m}N)(-1) & \to & (M/\mathfrak{m}N)(-1) & \to & (M/N)(-1) & \to & 0 \\ & & {\times z}\downarrow & & {\times z}\downarrow & \swarrow_{\varphi^{(z)}} & {\times z}\downarrow & & \\ 0 & \to & N/\mathfrak{m}N & \to & M/\mathfrak{m}N & \to & M/N & \to & 0 \end{array},$$

where $\times z$ is the multiplication map by z on each module, we have the following two exact sequence:

$$0 \to (N/\mathfrak{m}N)(-1) \to \left(0 \underset{M/\mathfrak{m}N}{:} z\right)(-1) \to \mathrm{Ker}\left(\varphi^{(z)}\right)(-1) \to 0,$$

$$0 \to \mathrm{Ker}\left(\varphi^{(z)}\right)(-1) \to \left(0 \underset{M/N}{:} z\right)(-1) \to M/(\mathfrak{m}N+zM) \to (N+zM)/(\mathfrak{m}N+zM) \to 0.$$

From the above exact sequences, (ii), i.e., $\mathrm{Ker}\left(\varphi^{(z)}\right) = 0$, is equivalent to each one of conditions from (iii) to (vi). □

Remark 3.5. If N is $\mathfrak{m}$-full in M w.r.t. z, then the following hold:

(1) We can assume $\left(0 \underset{M/N}{:} z\right)(-1) \subseteq N/\mathfrak{m}N$, i.e., there is a degree preserving injective morphism: $\iota : \left(0 \underset{M/N}{:} z\right)(-1) \to N/\mathfrak{m}N$ from the Ker-Coker sequence of the above diagram in the proof of Proposition 3.4.

(2) $0 \underset{M/N}{:} z = 0 \underset{M/N}{:} \mathfrak{m}$ since $0 \underset{M/N}{:} \mathfrak{m} \subseteq 0 \underset{M/N}{:} z \subseteq (N/\mathfrak{m}N)(1) \subseteq 0 \underset{M/N}{:} \mathfrak{m}$.

Theorem 3.6. *Let* $N \subseteq M \in \mathcal{A}$ *and* $L := \mathfrak{m}N \underset{M}{:} \mathfrak{m}^\infty = N \underset{M}{:} \mathfrak{m}^\infty$. *Then the following hold:*

(1) *If* $\mathfrak{m}\text{-full}(N;M) \neq \varnothing$, *then* $\mathfrak{m}\text{-full}(N;M) = \Lambda(M/\mathfrak{m}N) = \Lambda(L/\mathfrak{m}N) \cap \mathrm{NZD}_1(M/L)$, *especially* $\mathfrak{m}\text{-full}(N;M)$ *is a Zariski open subset of* R_1.

(2) *The following are equivalent:*
  (i) N *is* $\mathfrak{m}$*-full in* M;
  (ii) N *is* $\mathfrak{m}$*-full in* L.

Proof. (1) If $z \in \mathfrak{m}\text{-full}(N;M)$, then by Proposition 3.4, $l\left(0 \underset{M/\mathfrak{m}N}{:} z\right) = l(N/\mathfrak{m}N) = l\left(0 \underset{M/\mathfrak{m}N}{:} \mathfrak{m}\right)$. This implies $z \in \Lambda(M/\mathfrak{m}N)$. Hence $\mathfrak{m}\text{-full}(N;M) \subseteq \Lambda(M/\mathfrak{m}N)$. On the other hand, since $\mathfrak{m}\text{-full}(N;M) \neq \varnothing$, fix an element $z_0 \in \mathfrak{m}\text{-full}(N;M)$. If $z \in \Lambda(M/\mathfrak{m}N)$, then $l\left(0 \underset{M/\mathfrak{m}N}{:} z\right) = \lambda(M/\mathfrak{m}N) = l\left(0 \underset{M/\mathfrak{m}N}{:} z_0\right) = l(N/\mathfrak{m}N)$. By Proposition 3.4 (iii), this implies $z \in \mathfrak{m}\text{-full}(N;M)$. Hence $\Lambda(M/\mathfrak{m}N) \subseteq \mathfrak{m}\text{-full}(N;M)$. So we have first equality $\Lambda(M/\mathfrak{m}N) = \mathfrak{m}\text{-full}(N;M)$. Applying Lemma 2.7 (6), we also have $\Lambda(M/\mathfrak{m}N) = \Lambda(L/\mathfrak{m}N) \cap \mathrm{NZD}_1(M/L)$ a *Zariski open subset of* R_1.

(2) Applying (1), first we remark that if $\mathfrak{m}\text{-full}(N;L) \neq \varnothing$, then

$$\mathfrak{m}\text{-full}(N;L) = \Lambda(L/\mathfrak{m}N) \cap \mathrm{NZD}_1(L/L) = \Lambda(L/\mathfrak{m}N).$$

Hence again by (1), $\mathfrak{m}\text{-full}(N;M) \neq \varnothing$ if and only if $\mathfrak{m}\text{-full}(N;L) \neq \varnothing$ since both $\mathfrak{m}\text{-full}(N;L)$ and $\mathrm{NZD}_1(M/L)$ are Zariski open subsets in R_1 and $\mathrm{NZD}_1(M/L) \neq \varnothing$ by Lemma 2.7 (2). This implies N is $\mathfrak{m}$-full in M if and only if N is $\mathfrak{m}$-full in L. □

Remark 3.7. The field K being infinite is a crucial condition for Theorem 3.6., otherewise R_1 is a finite set, hence its Zariski topology is discrete and $\mathrm{NZD}_1(M/L)$ or $\Lambda(L/\mathfrak{m}N) \cap \mathrm{NZD}_1(M/L)$ may happen to be empty where $L = N \underset{M}{:} \mathfrak{m}^\infty$.

4. Componentwise $\mathfrak{m}$-fullness

Definition 4.1. *Let* $N \subseteq M \in \mathcal{A}$. N is called *componentwise* $\mathfrak{m}$*-full in* M if $N_{\langle i \rangle}$ is $\mathfrak{m}$-full in M for all $i \in \mathbb{Z}$.

Proposition 4.2. *Let* $N \subseteq M \in \mathcal{A}$. *If* N *is componentwise* $\mathfrak{m}$*-full in* M, *then* N *is* $\mathfrak{m}$*-full in* M.

Proof. If $N = 0$, the assertion clearly holds, so we assume $N \neq 0$. We denote $N_i := N_{\langle d_i \rangle}$ $(i = 1, \cdots, r)$ where $\mathrm{dnz}(N) = \{d_1 < \cdots < d_r\}$, then $N = N_1 + \cdots N_r$. Since each N_i $(i = 1, \cdots, r)$ is $\mathfrak{m}$-full in M, we can take an element $z \in \bigcap_{i=1}^{r} \mathfrak{m}\text{-full}(N_i;M) \neq \varnothing$ by Theorem (1). It is enough to show that $\mathfrak{m}N : z \subseteq N$. If $\xi \in \mathfrak{m}N : z$ is a homogeneous element, then $z\xi \in \mathfrak{m}N = \mathfrak{m}N_1 + \cdots + \mathfrak{m}N_r$. This implies $z\xi \in \mathfrak{m}N_p$ for some $1 \le p \le r$. Since N_p is $\mathfrak{m}$-full in M w.r.t. z and $\mathfrak{m}N : z$ is a graded module, we have $z \in N_p \subseteq N$, so $\mathfrak{m}N : z \subseteq N$. □

Lemma 4.3. *Let* $M \in \mathcal{A}$. *If* $z \in \mathrm{NZD}_1 M$, *then* $M_{\ge i+1} \underset{M}{:} z = M_{\ge i}$ *for any* $i \in \mathbb{Z}$.

Proof. $zM_{\ge i} \subseteq M_{\ge i+1}$, hence $M_{\ge i} \subseteq M_{\ge i+1} : z$. On the other hand, for any nonzero homogeneous element $0 \neq \xi \in M_{\ge i+1} : z$, $0 \neq z\xi \in M_{\ge i+1}$ and $\deg(z\xi) = 1 + \deg(\xi) \ge i+1$. This implies $\xi \in M_{\ge i}$. Since $M_{\ge i+1} : z$ is a graded module, we have $M_{\geq i+1} : z \subseteq M_{\geq i}$. We are done. □

Lemma 4.4. *Let* $N \subseteq M \in \mathcal{A}$ *. If* $l(M/N) < \infty$ *and* $0 \neq N = N_{\langle d \rangle}$ *, then the following hold:*

$$\mathfrak{m}^i N = M_{\geq i+d} \text{ for any integer } i \geq \sigma(M/N) - d.$$

Proof. Since $i + d \geq \sigma(M/N)$, we have $M_{\geq i+d} = N_{\geq i+d} = \mathfrak{m}^i N_{\langle d \rangle} = \mathfrak{m}^i N$. □

Definition 4.5. *Let* $N \subseteq M \in \mathcal{A}$. $\delta_M(0) := 0$, $\delta_M(N) := \max\{0, \sigma(M/N) - d\}$ if $0 \neq N = N_{\langle d \rangle}$ and for general $0 \neq N$, we define $\delta_M(N) := \max\left\{\delta_M\left(N_{\langle i \rangle}\right) \middle| i \in \mathrm{D}(N)\right\}$. We call $\delta_M(N)$ the *componentwise* $\mathfrak{m}$*-full defect of* N *in* M if $\operatorname{depth} M \geq 1$.

Remark 4.6. Let $N \subseteq M \in \mathcal{A}$ and $N_i := N_{\langle d_i \rangle}$, $L_i := N_i \underset{M}{:} \mathfrak{m}^\infty$ $(i = 1, \cdots, r)$ where $\mathrm{D}(N) = \{d_1 < \cdots < d_r\}$. Then $\delta_M(N) = 0$ if and only if $\delta_M(N_i) = 0$ for all $1 \leq i \leq r$. Moreover, for any $1 \leq i \leq r$, $\delta_M(N_i) = \delta_{L_i}(N_i)$ since $\sigma(M/N_i) = \sigma(L_i/N_i)$.

Lemma 4.7. *Let* $N \subseteq M \in \mathcal{A}$ *and* $L := N \underset{M}{:} \mathfrak{m}^\infty$ *. Assume* $\operatorname{depth} M \geq 1$*. If* $0 \neq N = N_{\langle d \rangle}$ *or* $N = 0$*, then the following hold:*

(1) *The following are equivalent:*
  (i) N *is* $\mathfrak{m}$*-full in* M *;*
  (ii) $\left(0 \underset{M/N}{:} \mathfrak{m}\right)_{\geq d} = 0$ *;*
  (iii) $d \geq \sigma(M/N)$*;*
  (iv) $N = L_{\geq d}$ *(Especially, in this case,* $L_{\geq d} = L_{\langle d \rangle}$*);*
  (v) $\mathrm{NZD}_1(M/N)_{\geq d} \neq \varnothing$.
  (vi) $d = \sigma(M/N)$ *if* $(0 \neq) N \neq L$ *,* $N = L$ *, i.e.,* $\sigma(M/N) = -\infty$ *or* $N = 0$ *;*
  (vii) $\delta_M(N) = 0$.

(2) $\mathrm{NZD}_1 M \cap \mathrm{NZD}_1(M/N)_{\geq d} \subseteq \mathfrak{m}\text{-full}(N; M)$.

(3) *If* N *is* $\mathfrak{m}$*-full in* M *, then* $N_{\langle i \rangle}$ *is* $\mathfrak{m}$*-full in* M *for all* $i \in \mathbb{Z}$ and $\mathfrak{m}^j N$ *is* $\mathfrak{m}$*-full in* M *for all* $j \in \mathbb{Z}_{\geq 0}$.

Proof. (1) If $N = 0$, then conditions from (i) to (vii) are all fulfiled. Hence we can assume $N \neq 0$.

$(\mathrm{i}) \Rightarrow (\mathrm{ii})$: By Remark 3.5 (1), we can assume $0 \underset{M/N}{:} \mathfrak{m} \subseteq (N/\mathfrak{m}N)(1)$. Hence $\left(0 \underset{M/N}{:} \mathfrak{m}\right)_{\geq d} = 0$.

$(\mathrm{ii}) \Leftrightarrow (\mathrm{iii})$: $\left(0 \underset{M/N}{:} \mathfrak{m}\right)_{\geq d} = 0$ if and only if $d > \operatorname{top}\left(0 \underset{M/N}{:} \mathfrak{m}\right)$ if and only if $d \geq 1 + \operatorname{top}\left(0 \underset{M/N}{:} \mathfrak{m}\right) = \sigma(M/N)$.

$(\mathrm{iii}) \Leftrightarrow (\mathrm{iv})$: Since $\sigma(M/N) = 1 + \operatorname{top}\left(0 \underset{M/N}{:} \mathfrak{m}\right) = 1 + \operatorname{top}\left(0 \underset{L/N}{:} \mathfrak{m}\right) = \sigma(L/N)$, $d \geq \sigma(M/N)$ if and only if $d \geq \sigma(L/N)$ if and only if $N = N_{\geq d} = L_{\geq d}$. Especially, in this case, $L_{\geq d} = L_{\langle d \rangle}$ since $N = N_{\langle d \rangle} = L_{\geq d}$.

$(\mathrm{ii}) \Leftrightarrow (\mathrm{v})$: Since $\left(0 \underset{M/N}{:} \mathfrak{m}\right)_{\geq d} = 0 \underset{(M/N)_{\geq d}}{:} \mathfrak{m}$, $\left(0 \underset{M/N}{:} \mathfrak{m}\right)_{\geq d} = 0$ if and only if $\mathrm{NZD}_1(M/N)_{\geq d} \neq \varnothing$ by Lemma 2.4.

$(\mathrm{iii}) \Leftrightarrow (\mathrm{vi})$:Since $\operatorname{depth} M \geq 1$, by Remark, $d \leq \sigma(M/N)$ holds if $(0 \neq) N \neq L$. Hence $d \geq \sigma(M/N)$ if and only if $d = \sigma(M/N)$ and $(0 \neq) N \neq M$.

$(\mathrm{iii}) \Leftrightarrow (\mathrm{vi})$:This directly follows form the definition of $\delta_M(N)$

$(\mathrm{iv}) \Rightarrow (\mathrm{i})$: We have already seen (iv) implies (v), hence $\mathrm{NZD}_1 M \cap \mathrm{NZD}_1(M/N)_{\geq d} \neq \varnothing$. For any element $z \in \mathrm{NZD}_1 M \cap \mathrm{NZD}_1(M/N)_{\geq d}$, $z \in \mathrm{NZD}_1 L$ and $l\left(\mathfrak{m}N \underset{M}{:} z\right) < l\left(N \underset{M}{:} z\right) < \infty$. Since $L = \mathfrak{m}N \underset{M}{:} \mathfrak{m}^\infty$, we have

$\mathfrak{m}N \underset{M}{:} z = \mathfrak{m}N \underset{L}{:} z = \mathfrak{m}L_{\langle d\rangle} \underset{L}{:} z = L_{\geq d+1} \underset{L}{:} z = L_{\geq d} = N$ by Lemma4.3 .

(2) In the proof of $(\mathrm{v}) \Rightarrow (\mathrm{i})$, we have already shown that $\mathfrak{m}N \underset{M}{:} z = N$ for all $z \in \mathrm{NZD}_1 M \cap \mathrm{NZD}_1 (M/N)_{\geq d}$.

(3) If $i < d$, then $N_{\langle i\rangle} = 0$ is $\mathfrak{m}$ -full in M w.r.t. $z \in \mathrm{NZD}_1 M \neq \varnothing$ since $\mathrm{depth} M \geq 1$. If $i \geq d$, then $\left(M/N_{\langle i\rangle}\right)_{\geq i} = (M/N_{\geq i})_{\geq i} = (M/N)_{\geq i} \subseteq (M/N)_{\geq d}$ and $\mathrm{NZD}_1 (M/N)_{\geq i} \supseteq \mathrm{NZD}_1 (M/N)_{\geq d} \neq \varnothing$. Hence $N_{\langle i\rangle}$ is $\mathfrak{m}$ -full in M by (vi) of (1). Similarly, $\mathfrak{m}^j N = N_{\langle d+j\rangle}$ is $\mathfrak{m}$ -full in M if $N \neq 0$. We are done. □

Remark 4.8. The equivalences among these, (ii),(iii),(iv) and (v) in Lemma 4.7 (1), hold without the assumption $\mathrm{depth} M \geq 1$.

Lemma 4.9. *Let* $N \subseteq M \in \mathcal{A}$. *If* $\mathrm{depth} M \geq 1$, *then the following hold:*

(1) *If* $0 \neq N = N_{\langle d\rangle}$, *then* $\delta_M\left(\mathfrak{m}^j N\right) = \max\{0, \delta_M(N) - j\}$ *for any integer* $j \geq 0$.

(2) $\delta_M\left(N_{\langle j\rangle}\right) \leq \delta_M(N)$ for any $j \in \mathbb{Z}$.

(3) *If* $\mathrm{D}(N) \subseteq \Phi \subseteq \mathbb{Z}$, *then* $\delta_M(N) = \max\left\{\delta_M\left(N_{\langle i\rangle}\right) \middle| i \in \Phi\right\}$.

(4) $\delta_M\left(\mathfrak{m}^j N\right) = \max\{0, \delta_M(N) - j\}$ *for any integer* $j \geq 0$.

Proof. (1) We prove this by induction on $j \geq 0$. If $j = 0$, clearly (1) holds. If $j > 0$, then from the short exact sequence: $0 \to \mathfrak{m}^{j-1}N/\mathfrak{m}^j N \to L/\mathfrak{m}^j N \to L/\mathfrak{m}^{j-1}N \to 0$, where $L := N \underset{M}{:} \mathfrak{m}^\infty$, we have:

$$\begin{aligned}
\delta_M\left(\mathfrak{m}^j N\right) &= \delta_L\left(\mathfrak{m}^j N\right) = 1 + \mathrm{top}\left(L/\mathfrak{m}^j N\right) - (d+j) \\
&= \max\left\{1 + \mathrm{top}\left(\mathfrak{m}^{j-1}N/\mathfrak{m}^j N\right) - (d+j), 1 + \mathrm{top}\left(L/\mathfrak{m}^{j-1}N\right) - (d+j)\right\} \\
&= \max\left\{0, \delta_M\left(\mathfrak{m}^{j-1}N\right) - 1\right\} \quad \left(1 + \mathrm{top}\left(\mathfrak{m}^{j-1}N/\mathfrak{m}^j N\right) - (d+j) = 0\right) \\
&= \max\{0, \max\{0, \delta_M(N) - (j-1)\} - 1\} \quad (\text{ the induction hypothesis}) \\
&= \max\{0, -1, \delta_M(N) - j\} = \max\{0, \delta_M(N) - j\}.
\end{aligned}$$

(2) If $N_{\langle j\rangle} = 0$, the assertion clearly holds, so we assume $N_{\langle j\rangle} \neq 0$, there exists $i \in \mathrm{D}(N)$ with $j \geq i$ such that $N_{\langle j\rangle} = \mathfrak{m}^{j-i} N_{\langle i\rangle}$. By (1), we have $\delta_M\left(N_{\langle j\rangle}\right) = \delta_M\left(\mathfrak{m}^{j-i}N_{\langle i\rangle}\right) = \max\left\{0, \delta_M\left(N_{\langle i\rangle}\right) - (j-i)\right\} \leq \delta_M(N)$.

(3) $\delta_M(N) = \max\left\{\delta_M\left(N_{\langle i\rangle}\right) \middle| i \in \mathrm{D}(N)\right\} \leq \max\left\{\delta_M\left(N_{\langle i\rangle}\right) \middle| i \in \Phi\right\}$ since $\mathrm{dnz}(N) \subseteq \Phi \subseteq \mathbb{Z}$. On the other hand, we have $\max\left\{\delta_M\left(N_{\langle i\rangle}\right) \middle| i \in \Phi\right\} \leq \delta_M(N)$ from (2). Hence $\delta_M(N) = \max\left\{\delta_M\left(N_{\langle i\rangle}\right) \middle| i \in \Phi\right\}$.

(4) Since $\mathrm{D}\left(\mathfrak{m}^j N\right) \subseteq \{i + j \mid i \in \mathrm{D}(N)\}$, $\left(\mathfrak{m}^j N_{\langle i\rangle}\right)_{\langle i+j\rangle} = \mathfrak{m}^j N_{\langle i\rangle}$ and by (1) $\delta_M\left(\mathfrak{m}^j N_{\langle i\rangle}\right) = \max\left\{0, \delta_M\left(N_{\langle i\rangle}\right) - j\right\}$ for any integer $j \geq 0$, we have

$$\begin{aligned}
\delta_M\left(\mathfrak{m}^j N\right) &= \max\left\{\delta_M\left(\left(\mathfrak{m}^j N_{\langle i\rangle}\right)_{\langle i+j\rangle}\right) \middle| i \in \mathrm{D}(N)\right\} \\
&= \max\left\{\delta_M\left(\mathfrak{m}^j N_{\langle i\rangle}\right) \middle| i \in \mathrm{D}(N)\right\}
\end{aligned}$$

$$= \max\left\{0, \max\left\{\delta_M\left(N_{\langle i\rangle}\right) \middle| i \in \mathrm{D}(N)\right\} - j\right\} = \max\{0, \delta_M(N) - j\}. \quad \square$$

Theorem 4.10. *Let* $0 \neq N \subseteq M \in \mathcal{A}$ *and* $N_i := N_{\langle d_i\rangle}$, $L_i := N_i \underset{M}{:} \mathfrak{m}^\infty$ $(i = 1, \cdots, r)$ *where* $\mathrm{D}(N) = \{d_1 < \cdots < d_r\}$. *If* $\mathrm{depth}M \geq 1$*, then the following are equivalent:*

(i) N *is componentwise* $\mathfrak{m}$*-full in* M*;*
(ii) N_i *is* $\mathfrak{m}$*-full in* M *for all* $1 \leq i \leq r$*;*
(iii) N_i *is* $\mathfrak{m}$*-full in* L_i *for all* $1 \leq i \leq r$*;*
(iv) $N_i = (L_i)_{\geq d_i}$ *for all* $1 \leq i \leq r$*;*
(v) $\delta_M(N_i) = 0$ *for all* $1 \leq i \leq r$*;*
(vi) $\delta_M(N) = 0$*;*
(vii) $\bigcap_{i=1}^{r} \mathrm{NZD}_1(M / N_i)_{\geq d_i} \neq \varnothing$*;*
(viii) $N = (L_1)_{\geq d_1} + \cdots + (L_r)_{\geq d_r}$ *and* $d_i = d'(L_i)$ *for all* $1 \leq i \leq r$.

Proof. $(\mathrm{i}) \Rightarrow (\mathrm{ii})$:This is clear by definition.
$(\mathrm{ii}) \Rightarrow (\mathrm{i})$:If $j < d_1$, then $N_{\langle j\rangle} = 0$ *is* $\mathfrak{m}$-full in M by Remark, since $\mathrm{depth}M \geq 1$. If $d_i \leq j < d_{i+1}$ for some $1 \leq i \leq r-1$ or $d_r \leq j$, then $N_{\langle j\rangle} = \mathfrak{m}^{j-d_i} N_i$ $1 \leq i \leq r-1$ or $N_{\langle j\rangle} = \mathfrak{m}^{j-d_r} N_r$. In each case, $N_{\langle j\rangle}$ is $\mathfrak{m}$-full in M by Lemma 4.7 (3).
$(\mathrm{ii}) \Leftrightarrow (\mathrm{iii}) \Leftrightarrow (\mathrm{iv}) \Leftrightarrow (\mathrm{v}) \Leftrightarrow (\mathrm{vi}) \Leftrightarrow (\mathrm{vii})$:These follows from Theorem Lemma and Remark.
$(\mathrm{iv}) \Rightarrow (\mathrm{viii})$: $N = N_1 + \cdots + N_r = (L_1)_{\geq d_1} + \cdots + (L_r)_{\geq d_r}$.
$(\mathrm{viii}) \Rightarrow (\mathrm{iv})$: First, we remark that $L_i = (N_1 + \cdots + N_i) \underset{M}{:} \mathfrak{m}^\infty$ since $l((N_1 + \cdots + N_i)/N_i) < \infty$ for $i = 1, \cdots, r$, so $L_i = N_{\langle d_1\rangle} + \cdots + N_{\langle d_i\rangle} \underset{M}{:} \mathfrak{m}^\infty \subseteq L_{i+1} = N_{\langle d_1\rangle} + \cdots + N_{\langle d_{i+1}\rangle} \underset{M}{:} \mathfrak{m}^\infty$ for $i = 1, \cdots, r-1$. Therefore, for any integer $1 \leq i \leq r$, we have $N_{d_i} = \left((L_1)_{\geq d_1} + \cdots + (L_i)_{\geq d_i} + \cdots + (L_r)_{\geq d_r}\right)_{d_i} = (L_1)_{d_i} + \cdots + (L_i)_{d_i} = (L_i)_{d_i}$. This implies $N_{\langle d_i\rangle} = (L_i)_{\langle d_i\rangle} = (L_i)_{\geq d_i}$ since $d_i = d'(L_i)$ for $1 \leq i \leq r$. $\square$

Remark 4.11. If N is componentwise $\mathfrak{m}$-full in M and $\mathrm{depth}M \geq 1$, then from Theorem4.10 and its proof,

$$\varnothing \neq \mathrm{NZD}_1(M) \cap \bigcap_{i=1}^{r} \mathrm{NZD}_1(M / L_i) \subseteq \mathrm{NZD}_1(M) \cap \bigcap_{i=1}^{r} \mathrm{NZD}_1(M / N_i)_{\geq d_i} \subseteq \bigcap_{j \in \mathbb{Z}} \mathfrak{m}\text{-full}\left(N_{\langle j\rangle}; M\right).$$

Corollary 4.12. *Let* $N \subseteq M \in \mathcal{A}$. *If* $\mathrm{depth}M \geq 1$*, then* $\mathfrak{m}^j N$ *is componentwise* $\mathfrak{m}$*-full in* M *for any integer* $j \geq \delta_M(N)$.

Proof. If $N = 0$, the assertion clearly holds, so we assume $N \neq 0$. By Lemma 4.9 (4) and $j \geq \delta_M(N)$, we have $\delta_M(\mathfrak{m}^j N) = \max\{0, \delta_M(N) - j\} = 0$. Hence by Theorem 1.10 (vi), $\mathfrak{m}^j N$ is componentwise $\mathfrak{m}$-full in M. $\square$

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