Paper

On Some Diophantine Equations (II)

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Abstract In this paper, we treat the equation $(p^{x_{12}}-1)/q^{y_1}=(q^{y_{12}}-1)/p^{x_1}=k$, where k is a fixed integer. Especially, we study the cases of k satisfied with $1\le k\le 5$ or prime $k\ge 7$. The equation $(p^{x_{12}}-1)/q^{y_1}=(q^{y_{12}}-1)/p^{x_1}=k$ has solutions in the cases of $k=1,\ 3,\ 5$.

Keywords : Diophantine equation, Existence condition of solution, Quadric field, Fundamental unit, Residue class

1. Intoroduction

Let $a,\ b,\ x,\ y$ be positive integers. The diophantine equation $a^x-b^y=c$, where c is a fixed nonzero integer, has been treated by many authors. In the case of $c=1$, the following Catalan's theorem[(1)] is well known:

Catalan's theorem Let $a,\ b,\ x,\ y>1$. Then $a^x-b^y=1$ has a unique solution $(a,\ b,\ x,\ y)=(3,\ 2,\ 2,\ 3)$.

M.A.Bennett[(2)] proved, if $a,\ b>1$, $a^x-b^y=c$ has at most two solutions in x and y. We can find that two solutions of $a^x-b^y=c$ in x and y, corresponding to the following set of equations:

$$3^1-2^1=3^2-2^3=1;\qquad 2^3-3^1=2^5-3^3=5;\qquad 2^4-3^1=2^8-3^5=13;$$

$$2^3-5^1=2^7-5^3=3;\qquad 13^1-3^1=13^3-3^7=10;$$

$$91^1-2^1=91^2-2^{13}=89;\qquad 6^1-2^1=6^2-2^5=4;\qquad 15^1-6^1=15^2-6^3=9; \tag{1.2}$$

$$280^1-5^1=280^2-5^7=275;\qquad 4930^1-30^1=4930^2-30^5=4900;$$

$$6^4-3^4=6^5-3^8=1215.$$

Furthermore, he referred the following conjecture:

Conjecture If $a,\ b>1$ and $c>0$, then $a^x-b^y=c$ has at most one solution in x and y, except for those triples $(a,\ b,\ c)$ corresponding to (1.2).

Let $p,\ q$ be primes with $p<q$. In this paper, we treat the equation $p^{x_1}-q^{y_1}=p^{x_2}-q^{y_2}=c$, where $x_1,\ x_2\ y_{,1}\ y_2$

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are positive integers with $x_1 < x_2$ and $y_1 < y_2$. Now, let $x_{12} = x_2 - x_1$, $y_{12} = y_2 - y_1$. Then $p^{x_1} - q^{y_1} = p^{x_2} - q^{y_2}$ leads

(1.3) $$\frac{p^{x_{12}} - 1}{q^{y_1}} = \frac{q^{y_{12}} - 1}{p^{x_1}} := k \ (\in \mathbb{N}).$$

2. Equation $\dfrac{p^{x_{12}} - 1}{q^{y_1}} = \dfrac{q^{y_{12}} - 1}{p^{x_1}} = k$

Let p and q be primes with $p < q$. Let x_1, y_1, x_{12} and y_{12} be positive integers. In this section, we consider the equation

(2.1) $$\frac{p^{x_{12}} - 1}{q^{y_1}} = \frac{q^{y_{12}} - 1}{p^{x_1}} = k,$$

where k is a fixed positive integer. Then, since $p < q$, it follows $x_{12} > 1$ and $q \nmid p - 1$. Here, we note as follows:

If k is odd then $p = 2$ and q is an odd prime;

If k is even then both p and q are odd primes.

2.1 The case of $k = 1$

Let $k = 1$. Then (2.1) leads

(2.1.1) $$2^{x_{12}} - q^{y_1} = 1$$

and

(2.1.2) $$q^{y_{12}} - 2^{x_1} = 1.$$

Theorem 1 The system of equations (2.1.1) and (2.1.2) has two solutions $(q, x_1, y_1, x_{12}, y_{12}) = (3, 3, 1, 2, 2)$, $(3, 1, 1, 2, 1)$.

Proof It is proved by using Catalan's theorem. Since $x_{12} > 1$, it follows $y_1 = 1$ from (2.1.1).

First, suppose $x_1 > 1$ and $y_{12} > 1$. Then, (2.1.2) has a unique solution $(q, x_1, y_{12}) = (3, 3, 2)$. Thus, from (2.1.1), $x_{12} = 2$ is obtained. Therefore $(3, 3, 1, 2, 2)$ is a solution of system of equations.

Next, suppose $x_1 = 1$ or $y_{12} = 1$. If $x_1 = 1$, from (2.1.2), then $q^{y_{12}} = 3$, and so $(q, y_{12}) = (3, 1)$. Thus, from (2.2.1), $y_1 = 1$ is obtained. If $y_{12} = 1$, by adding (2.1.1) and (2.1.2), then $2^{x_{12}} - 2^{x_x} = 2$. Thus $2^{x_{12}-1} - 2^{x_x - 1} = 1$, and so $(x_1, x_{12}) = (1, 2)$. Therefore, in each case, $(3, 1, 1, 2, 1)$ is a solution of equations.

□

2.2 The case of $k = 2$

Let $k = 2$. Then (2.1) leads

(2.2.1) $$\frac{p-1}{2}\cdot\frac{p^{x_{12}}-1}{q^{y_1}(p-1)}=\frac{q-1}{2p^s}\cdot\frac{q^{y_{12}}-1}{p^{x_1-s}(q-1)}=1,$$

where $s=\nu_p(q-1)$.

Then we have $(p-1)/2=1$, and so $p=3$. Thus $(3^{x_{12}}-1)/2q^{y_1}=1$ and so $3^{x_{12}}-3=2(q^{y_1}-1)$. If $s>1$, since $x_{12}>1$, then $-3\equiv 0 \pmod 9$. We have a contradiction. If $s=0$ then $(q-1)/2=1$, and so $q=3$. This is a contradiction to $p<q$. Therefore, from now on, we can suppose $s=1$. And, it follows $(q-1)/6=1$, and so $q=7$.

Then (2.2.1) leads

(2.2.2) $$3^{x_{12}}-1=2\cdot 7^{y_1}$$

and

(2.2.3) $$7^{y_{12}}-1=3^{x_1-1}.$$

Proposition 1 Let d be a square free integer. Let $\zeta=\alpha+\beta\sqrt{d}$ be a element of quadric field $\mathbb{Q}(\sqrt{d})$. Put $\zeta^n=\alpha_n+\beta_n\sqrt{d}$ for $n\in\mathbb{Z}$. Then both α_n and β_n are satisfied with the recurrence formula $X_{n+2}=2\alpha X_{n+1}-N(\zeta)X_n$.

Proof Since $\zeta^2=\alpha^2+\beta^2 d+2\alpha\beta\sqrt{d}=\alpha^2+\beta^2 d+2\alpha(\zeta-\alpha)$, it follows $\zeta^2=2\alpha\zeta-N(\zeta)$. Thus $\zeta^{n+2}=2\alpha\zeta^{n+1}-N(\zeta)\zeta^n$, and so $\alpha_{n+2}+\beta_{n+2}\sqrt{d}=2\alpha(\alpha_{n+1}+\beta_{n+1}\sqrt{d})-N(\zeta)(\alpha_n+\beta_n\sqrt{d})$. Therefore the proof is complete.

□

Theorem 2 The equation (2.2.2) has no solutions.

Proof The equation (2.2.2) gives $x_{12}\equiv 1 \pmod 2$. Suppose $y_{12}\equiv 1 \pmod 2$ is odd. It follows $\nu_2(7^{y_1}-1)=1$. Then, from (2.2.2), $2=\nu_2(2(7^{y_1}-1))=\nu_2(3(3^{x_{12}}-1))>2$ is obtained. We have a contradiction.

Suppose $y_{12}\equiv 0 \pmod 2$. Let $x_{12}=2x+1$ and $y_1=2y$ ($x,\ y\in\mathbb{N}$). Then (2.2.2) leads

(2.2.4) $$3^{2x+2}-6\cdot 7^{2y}=3.$$

Let $\varepsilon=5+2\sqrt{6}$ be the fundamental unit of $\mathbb{Q}(\sqrt{6})$. Put $\varepsilon^n=t_n+u_n\sqrt{6}$ for $n\in\mathbb{Z}$. Then, there exists $N\in\mathbb{N}$ such that $3^{x+1}+7^{y_1}\sqrt{6}=(3+\sqrt{6})\varepsilon^N=(3t_N+6u_N)+(t_N+3u_N)\sqrt{6}$. Furthermore, put $v_n=t_n+3u_n$. Then sequence $\{v_n\}$ is satisfied with $v_0=1,\ v_1=11$ and $v_{n+2}=10v_{n+1}-v_n$. That is $\cdots,\ 1,\ 11,\ 109,\ 1079,\ 10681,\ \cdots$. Since $v_{n+2}=10v_{n+1}-v_n$, it follows $v_{n+4}\equiv 3v_{n+3}-v_{n+2}\equiv v_{n+2}-3v_{n+1}\equiv -v_n \pmod 7$. Thus $7\nmid v_n$. This is a contradiction to $v_N=t_N+3u_N=7^{y_1}$.

Therefore (2.2.2) has no solutions.

□

2.2 The case of $k=3$

Let $k=3$. Then (2.1) leads

(2.3.1) $$2^{x_{12}}-1=3\cdot q^{y_1}$$

and

(2.3.2) $\quad \dfrac{q-1}{2^s}\cdot\dfrac{q^{y_{12}}-1}{2^{x_1-s}(q-1)}=3$,

where $s=\nu_2(q-1)$.

The equation (2.3.1) gives $x_{12}\equiv 0 \pmod 2$. Let $x_{12}=2x$ ($x\in\mathbb{N}$). If $x=1$ then $q^{y_1}=1$. We have a contradiction. Therefore, from now on, we can suppose $x\geq 2$. Then (2.3.1) leads $4(4^{x-2}+\cdots+1)=q^{y_1}-1$. Thus $\nu_2(q^{y_1}-1)=2$. Hence $y_1\equiv 1 \pmod 2$ and $s=2$. Furthermore, from (2.3.2), we have $(q-1)/2^2=1$ or 3, and so $q=5$ or 13.

Let $y_1=2y+1$ ($y\in\mathbb{N}\cup\{0\}$). Then (2.3.1) leads

(2.3.3) $\quad 2^{2x}-15\cdot 5^{2y}=1$

or

(2.3.4) $\quad 2^{2x}-39\cdot 13^{2y}=1$.

Proposition 2 The equation (2.3.4) has no solutions.

Proof Let $\varepsilon=25+4\sqrt{39}$ be the fundamental unit of $\mathbb{Q}(\sqrt{39})$. Put $\varepsilon^n=t_n+u_n\sqrt{39}$ for $n\in\mathbb{Z}$. Then, there exists $N\in\mathbb{N}$ such that $2^x+13^y\sqrt{39}=t_N+u_N\sqrt{39}$. Since t_n is satisfied with $t_0=1,\ t_1=25$ and $t_{n+2}=50t_{n+1}-t_n$, it follows $2\nmid t_n$. This is a contradiction to $t_N=2^x$. Thus (2.3.4) has no solutions.

□

Proposition 3 The equation (2.3.3) has a unique solution $(x,\ y)=(2,\ 0)$.

Proof Let $\varepsilon=4+\sqrt{15}$ be the fundamental unit of $\mathbb{Q}(\sqrt{15})$. Put $\varepsilon^n=t_n+u_n\sqrt{15}$ for $n\in\mathbb{Z}$. Then, there exists $N\in\mathbb{N}\cup\{0\}$ such that $2^x+5^y\sqrt{15}=t_N+u_N\sqrt{15}$. Furthermore, u_n is satisfied with $u_0=0,\ u_1=1$ and $u_{n+2}=8u_{n+1}-u_n$. That is $\cdots,\ 0,\ 1,\ 8,\ 63,\ 496,\ 3905,\ \cdots$. Thus we have

$$u_{n+5}\equiv 3u_{n+4}-u_{n+3}\equiv 3u_{n+3}-3u_{n+2}\equiv u_{n+2}-3u_{n+1}\equiv -u_n \pmod 5$$

and

$$u_{n+5}\equiv -3u_{n+4}-u_{n+3}\equiv -3u_{n+3}+3u_{n+2}\equiv u_{n+2}+3u_{n+1}\equiv -u_n \pmod{11}.$$

Therefore, for $n\in\mathbb{N}$, $5\,|\,u_n$ leads $5\,|\,n$. Furthermore $5\,|\,n$ leads $11\,|\,u_n$. Hence, $u_N=5^y$ leads $11\,|\,u_N$, if $N\in\mathbb{N}$. We have a contradiction. Thus $N=0$. Then we have $2^x=4$ and $5^y=1$. Therefore $(x,\ y)=(2,\ 0)$ is a solution of (2.3.3).

□

Theorem 3 The system of equations (2.3.1) and (2.3.2) has a unique solution $(q,\ x_1,\ y_1,\ x_{12},\ y_{12},\ s)=(5,\ 3,\ 1,\ 4,\ 2,\ 2)$.

Proof From proposition 2 and 3, it will be sufficient to prove that $(x_1,\ y_{12})=(3,\ 2)$ is a unique solution of (2.3.2) as $(q,\ s)=(5,\ 2)$. Then (2.3.2) leads

(2.3.5) $\quad 5^{y_{12}}-3\cdot 2^{x_1}=1$.

Since $3 \mid 5^{y_{12}} - 1$, it follows $y_{12} \equiv 0 \pmod 2$. Let $y_{12} = 2y'$ ($y \in \mathbb{N}$). From (2.3.5), We have

(2.3.6) $\quad 25^{y'-1} + \cdots + 1 = 2^{x_1 - 3}$

If y is even then $2^{x_1-3} = 25^{y'-1} + \cdots + 1 \equiv 0 \pmod{13}$. We have a contradiction. Thus y' is odd. Then, since $25^{y'-1} + \cdots + 1$ is odd, it follows $x_1 = 3$. Thus $y' = 1$, and so $y_{12} = 2$. Therefore $(3, 2)$ is a unique solution of (2.3.2).

□

2.4 The case of $k = 4$

Let $k = 4$. Then (2.1) leads

(2.4.1) $$\frac{p-1}{4} \cdot \frac{p^{x_{12}} - 1}{q^{y_1}(p-1)} = \frac{q-1}{4p^s} \cdot \frac{q^{y_{12}} - 1}{p^{x_1 - s}} = 1,$$

where $s = \nu_p(q-1)$.

Theorem 4 The equation (2.4.1) has no solutions.

Proof We have $(p-1)/4 = 1$, and so $p = 5$. Thus $(5^{x_{12}} - 1)/4q^{y_1} = 1$, and so $5^{x_{12}} - 4q^{y_1} = 1$. If $s > 1$, since $x_{12} > 1$, then $-5 \equiv 0 \pmod{25}$. We have a contradiction. If $s = 0$ then $(q-1)/4 = 1$, and so $q = 5$. This is a contradiction to $p < q$. Therefore, from now on, we can suppose $s = 1$. And, it follows $(q-1)/20 = 1$, and so $q = 21$. This is a contradiction to prime.
Thus (2.4.1) has no solutions.

□

2.5 The case of $k = 5$

Let $k = 5$. Then (2.1) leads

(2.5.1) $\quad 2^{x_{12}} - 1 = 5 \cdot q^{y_1}$,

and

(2.5.2) $\quad q^{y_{12}} - 1 = 5 \cdot 2^{x_1}$.

Theorem 5 The system of equations (2.5.1) and (2.5.2) has a unique solution $(q, x_1, y_1, x_{12}, y_{12}) = (3, 4, 1, 4, 4)$.

Proof The equation (2.5.1) gives $4 \mid x_{12}$. Let $x_{12} = 4x$ ($x \in \mathbb{N}$). Then $3(16^{x-1} + \cdots + 1) = q^{y_1}$ is obtained. Thus we have $q = 3$ and

(2.5.3) $\quad 16^{x-1} + \cdots + 1 = 3^{y_1 - 1}$.

If $x > 1$ then $x \equiv 0 \pmod 3$. Thus $16^2 + 16 + 1 \mid 16^{x-1} + \cdots + 1$. Since $16^2 + 16 + 1 = 273 = 3 \times 7 \times 13$, it follows $7 \mid 3^{y_1 - 1}$. We have a contradiction. Therefore $(x, y_1) = (1, 1)$ is a solution of (2.5.3).

Similarly, (2.5.2) gives $4 \mid y_{12}$. Let $y_{12} = 4x$ ($x \in \mathbb{N}$). Thus we have

(2.5.4) $\quad 81^{y-1}+\cdots+1=2^{x_1-4}$.

If $y>1$ then $y\equiv 0 \pmod 2$. Thus $81+1 \mid 81^{y-1}+\cdots+1$. Since $81+1=82=2\times 41$, it follows $41 \mid 2^{x_1-1}$. We have a contradiction. Therefore $(y,\ x_1)=(1,\ 1)$ has a solution of (2.5.4). And the proof is complete.

□

2.6 The case of prime $k\geq 7$

Let k be a fixed prime with $k\geq 7$. Then (2.1) leads

(2.6.1) $\quad 2^{x_{12}}-1=k\cdot q^{y_1}$,

and

(2.6.2) $\quad q^{y_{12}}-1=k\cdot 2^{x_1}$.

Proposition 4 If $q=3$ then the system of equations (2.6.1) and (2.6.2) has no solutions.

Proof Let $q=3$. Then, since $3 \mid 2^{x_{12}}-1$, it follows $x_{12}\equiv 0$ (mod 2). First, suppose $x_{12}\equiv 0$ (mod 4). Let $x_{12}=4x$ ($x\in\mathbb{N}$). Then, from (2.6.1), we have $(2^{2x}+1)\{(2^{2x}-1)/3^{y_1}\}=k$. Thus $(2^{2x}-1)/3^{y_1}=1$, and so $2^{2x}-3^{y_1}=1$. Therefore, by Catalan's theorem, we have $(x,\ y_1)=(1,\ 1)$. Hence $(x_{12},\ y_1)=(4,\ 1)$ is a solution of (2.6.1). Furthermore $k=5$ holds. This is a contradiction to $k\geq 7$. Next, suppose $x_{12}\equiv 0 \pmod 4$. Let $x_{12}=4x+2$ ($x\in\mathbb{N}\cup\{0\}$). Then, (2.6.1) leads, we have $\{(2^{2x+1}+1)/3^{y_1}\}(2^{2x+1}-1)=k$. We consider the following cases;

(A) $(2^{2x+1}+1)/3^{y_1}=k$ and $2^{2x+1}-1=1$;

(B) $(2^{2x+1}+1)/3^{y_1}=1$ and $2^{2x+1}-1=k$.

In the case (A), $2^{2x+1}-1=1$ give $x=0$. Thus $3/3^{y_1}=k$ is given. Hence $y_1=1$ and $k=1$ hold. This is a contradiction to $k\geq 7$. In the case(B), $(2^{2x+1}+1)/3^{y_1}=1$ leads $3^{y_1}-2^{2x+1}=1$. Thus, by Catalan's theorem, we have $(x,\ y_1)=(0,\ 1)$ and $(x,\ y_1)=(1,\ 2)$. When $(x,\ y_1)=(0,\ 1)$ holds, $k=1$ is given. This is a contradiction to $k\geq 7$. When $(x,\ y_1)=(1,\ 2)$ holds, $k=7$ is given. Thus $(x_{12},\ y_1,\ k)=(6,\ 2,\ 7)$ has a solution of (2.6.1).

Then, from (2.6.2), we have $7 \mid 3^{y_{12}}-1$. Hence $6 \mid y_{12}$. This result leads $3^6-1 \mid 3^{y_{12}}-1$. Since $3^6-1=728=2^3\times 7\times 13$, it follows $13 \mid 2^{x_1}$. We have a contradiction. Thus the system of equations (2.6.1) and (2.6.2) has no solutions.

□

From now on, we can suppose $q\geq 5$. Then $q\equiv \pm 1 \pmod 3$ holds. Therefore, if y_{12} is even then $3 \mid q^{y_{12}}-1$. Thus, from (2.6.2), we have $3 \mid k$. This is a contradiction to prime $k\geq 7$. Thus y_{12} is odd. Similarly, from (2.6.1), we can obtain that x_{12} is odd. Furthermore, we have $\left(\dfrac{2}{q}\right)=\left(\dfrac{2^{x_1}}{q}\right)=\left(\dfrac{(2^{x_1}-1)+1}{q}\right)=\left(\dfrac{1}{q}\right)=1$, where notation $\left(\dfrac{\cdot}{\cdot}\right)$ is Legendre's symbol. Thus $q\equiv \pm 1 \pmod 8$. Similary, $k\equiv \pm 1 \pmod 8$ is obtained.

Proposition 5 If $y_{12} > 1$ then the system of equations (2.6.1) and (2.6.2) has no solutions.

Proof The equation (2.6.2) leads $\{(q-1)/2^{x_1}\}\{(q^{y_{12}}-1)/(q-1)\} = k$. Since $(q-1)/2^{x_1} < (q^{y_{12}}-1)/(q-1)$, it follows $(q-1)/2^{x_1} = 1$. Hence $q = 2^{x_1}+1$. Since q is a prime with $q \geq 7$, it follows $x_1 \equiv 0 \pmod 2$. Then, from (2.6.1), we have $2^{x_1}+1 = q \mid 2^{x_{12}}-1$. We have a contradiction, because $x_1 \equiv 0 \pmod 2$ and $x_{12} \equiv 1 \pmod 2$ hold. Thus the system of equations (2.6.1) and (2.6.2) has no solutions.

□

Let $y_{12} = 1$. Then (2.6.2) leads

(2.6.3) $\quad q = k \cdot 2^{x_1} + 1$

Proposition 6 **4** If $x_1 = 1$ or $x_1 = 2$ then the system of equations (2.6.1) and (2.6.3) has no solutions.

Proof First, suppose $x_1 = 2$. (2.6.3) leads $q = 4k+1$. If $q \equiv 1 \pmod 8$ then $k \equiv 0 \pmod 2$. This is a contradiction to prime $k \geq 7$. If $q \equiv -1 \pmod 8$ then $2k \equiv -1 \pmod 4$. We have a contradiction.

Next. Suppose $x_1 = 1$. (2.6.3) leads $q = 2k+1$. If $q \equiv 1 \pmod 8$ then $k \equiv 0 \pmod 4$. This is a contradiction to prime $k \geq 7$. If $q \equiv -1 \pmod 8$ then $k \equiv -1 \pmod 4$. Since $k \equiv \pm 1 \pmod 8$, it follows $k \equiv -1 \pmod 8$. We note that x_{12} is odd with $x_{12} \geq 3$. Thus, from (2.6.1), we have $(-1)^{y_1} \equiv 1 \pmod 8$. Hence y_1 is even.

On the other hand, if $q \equiv 1$ then $k \equiv 0 \pmod 3$. This is a contradiction to prime $k \geq 7$. Thus $q \equiv -1 \pmod 3$. And it follows $k \equiv -1 \pmod 3$. Then, from (2.6.1), we have $1 \equiv (-1)^{y_1+1} \pmod 3$. Thus y_1 is odd. We have a contradiction, and the proof is complete.

□

Theorem 6 The system of equations (2.6.1) and (2.6.3) has no solutions.

Proof By proposition 6, it will be sufficient to prove this for the case where $x_1 \geq 3$. Then, (2.6.3) give $q \equiv 1 \pmod 8$. Furthermore, since x_{12} is odd with $x_{12} \geq 3$, (2.6.1) gives $k \equiv -1 \pmod 8$.

Now, let $O_q(2) = m$ ($m \cup \mathbb{N} \setminus \{1\}$). Then, there exists $s \in \mathbb{N}$ such that $q^s \mid 2^m - 1$ and $q^{s+1} \nmid 2^m - 1$. From (2.6.1), we have $m \mid x_{12}$. Thus m is odd with $m \geq 3$. If $2^m - 1 = q^s$ then $-1 \equiv 1 \pmod 8$. We have a contradiction. Thus $2^m - 1 = k \cdot q^s$. Furthermore, since $m \mid q-1$, (2.6.3) give $m \mid k \cdot 2^{x_1}$. Thus $m = k$. Then, since $k \cdot q^s = 2^k - 1 = 2(2^{k-1}-1)+1$, it follows $0 \equiv 1 \pmod k$. We have a contradiction, and the proof is complete.

□

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