Paper

On Some Diophantine Equations (I)

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Abstract In this paper, we consider two equations $(p^x-1)/(q-1)=(q^y-1)/(q-1)$ and $a^{x_1}-b^{y_1}=a^{x_2}-b^{y_2}=1$. Especially, on $(p^x-1)/(q-1)=(q^y-1)/(q-1)$, we study in the case of $y=3$ and in the case of $p=2$. And, without using Catalan's theorem, we prove that $a^{x_1}-b^{y_1}=a^{x_2}-b^{y_2}=1$ has a unique solution.

Keywords : Diophantine equation, Existence condition of solution, Quadric filed, Fundamental unit, Residue class

1. Introduction

Let a, b, x, y be positive integers. The diophantine equation $a^x-b^y=c$, where c is a fixed nonzero integer, has been treated by many authors. In the case of $c=1$, the following Catalan's theorem[(1)] is well known:

Catalan's theorem Let $a, b, x, y>1$. Then $a^x-b^y=1$ has a unique solution $(a, b, x, y)=(3, 2, 2, 3)$.

M.A.Bennett[(2)] proved the following theorem:

Theorem 1.1 if a, b and c are nonzero integers with $a, b\geq 2$, then the equation $a^x-b^y=c$ has at most two solutions in positive integers x and y.

R.Balasubramanian and T.N.Shorey[(3),(4)] treated the equation $a(x^m-1)/(x-1)=b(y^n-1)/(y-1)$. Y.Motoda[(5)] treated the equation $(p^{2e+1}+1)/(p+1)=(q^3+1)/(q+1)$, where p and q are distinct primes.

In this paper, we consider two equations $(p^x-1)/(p-1)=(q^y-1)/(q-1)$ and $a^{x_1}-b^{y_1}=a^{x_2}-b^{y_2}=1$. On $(p^x-1)/(p-1)=(q^y-1)/(q-1)$, we study it in the case of $y=3$ and in the case of $p=2$. In each case, $(p^x-1)/(p-1)=(q^y-1)/(q-1)$ has a unique solution $(p, q, x, y)=(2, 5, 5, 3)$ under some assumptions. Furthermore, the equation $a^{x_1}-b^{y_1}=a^{x_2}-b^{y_2}=1$ has a unique solution $(a, b, x_1, y_1, x_2, y_2)=(3, 2, 1, 1, 2, 3)$. This result is proved without using Catalan's theorem.

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2. Equation $\dfrac{p^x-1}{p-1}=\dfrac{q^y-1}{q-1}$

Let p and q be positive primes with $p \neq q$. Let x and y be positive integers with $x,\ y \geq 2$. In this section, we consider the equation

(2.1) $$\frac{p^x-1}{p-1}=\frac{q^y-1}{q-1}.$$

2.1 The case of $y=3$

Let $y=3$. Then (2.1) leads

(2.2) $$\frac{p^x-1}{p-1}=q^2+q+1.$$

Proposition 2.1 q is an odd prime.

Proof If $q=2$ then p is an odd prime. And, from (2.2), $p(7-p^{x-1})=6$ holds. Thus $p=3$ and $7-p^{x-1}=2$, and so $3^{x-1}=5$. Therefore we have a contradiction.

□

Proposition 2.2 x is odd.

Proof Suppose that x is even. If p is an odd prime then $(p^x-1)/(p-1)$ is even. This is a contradiction to proposition 2.1. From (2.2), If $p=2$ then $(q+1)/2=(2^{x-1}-1)/q$. Thus $(q+1)/2 \equiv 1 \pmod 2$, and so $q \equiv 1 \pmod 4$. Furthermore, since $q \mid 2^{x-1}-1$, it follows $\left(\frac{2}{q}\right)=\left(\frac{2^{x-1}}{q}\right)=\left(\frac{(2^{x-1}-1)+1}{q}\right)=\left(\frac{1}{q}\right)=1$, where $\left(\frac{\cdot}{\cdot}\right)$ is Legendre's symbol. This leads $q \equiv \pm 1 \pmod 8$. Since $q \equiv 1 \pmod 4$, $q \equiv 1 \pmod 8$ is obtained. Then (2.2) leads $2^x \equiv 4 \pmod 8$, and so $2^{x-2} \equiv 1 \pmod 2$. Thus $x=2$. By substituting this result for (2.2), we have $q^2+q-2=0$, and so $q=1,\ -2$. This is a contradiction to proposition 2.1.

□

Theorem 2.1 The equation (2.2) has a unique solution $(p,\ q,\ x)=(2,\ 5,\ 5)$.

Proof Let $x=2e+1$ ($e \in \mathbb{N}$). Then (2.2) leads

(2.3) $$\frac{p^{2e}-1}{q(p-1)}=\frac{q+1}{p}:=t(\in \mathbb{N}).$$

Then, since $q+1$ is even with $q+1 \geq 4$, it follows $t=(q+1)/p \neq 1$. Furthermore (2.3) leads

(2.4) $$(p^e+1)(p^e-1)=(pt-1)(pt-t).$$

Then, since $(pt-1)-(pt-t)=t-1 \geq 1$, $p^e-1<pt-t \leq pt-1<p^e+1$ doesn't occur. Thus, from (2.4), we have

(2.5) $$pt-t \leq p^e-1<p^e+1 \leq pt-1.$$

If $pt-t<p^e-1<p^e+1<pt-1$, since $q=pt-1$, then $(p^e-1)/q<(p^e+1)/q<1$. This is a contradiction to $(p^e-1)(p^e+1)/q\in\mathbb{N}$. Thus we have

$$(2.6)\qquad \begin{cases} pt-t=p^e-1 \\ pt-1=p^e+1 \end{cases}.$$

That is $t=3$, $p(3-p^{e-1})=2$, and so $p=2,\ e=2$. Therefore $(p,\ q,\ x)=(2,\ 5,\ 5)$ is a solution of (2.2).

□

2.2 The case of $p=2$ and $q\equiv 5\ (\mathrm{mod}\ 8)$

Let $p=2$. Then (2.1) leads

$$(2.7)\qquad 2^x-1=\frac{q^y-1}{q-1}.$$

We note that q is an odd prime. From (2.7), since 2^x-1 is odd, we have $y\equiv 1\ (\mathrm{mod}\ 2)$. Now let $y=2f+1$ ($f\in\mathbb{N}$). Then (2.7) leads

$$(2.8)\qquad \frac{2^{x-1}-1}{q}=\frac{q+1}{2}\cdot\frac{(q^2)^f-1}{q^2-1}(\in\mathbb{N}).$$

Since $(2^{x-1}-1)/q$ is odd, it follows that f is odd and $(q+1)/2\equiv 1\ (\mathrm{mod}\ 2)$, and so $q\equiv 1\ (\mathrm{mod}\ 4)$.

From now on, we suppose $q\equiv 5\ (\mathrm{mod}\ 8)$.

If x is even, from (2.8), then we have $\left(\frac{2}{q}\right)=\left(\frac{2^{x-1}}{q}\right)=\left(\frac{(2^{x-1}-1)+1}{q}\right)=\left(\frac{1}{q}\right)=1$. Thus $q\equiv\pm 1\ (\mathrm{mod}\ 8)$. Since $q\equiv 1\ (\mathrm{mod}\ 4)$, $q\equiv 1\ (\mathrm{mod}\ 8)$ holds. This is a contradiction to $q\equiv 5\ (\mathrm{mod}\ 8)$. Thus x is odd.

Let $x\equiv 2e+1$ ($e\in\mathbb{N}$). Then, (2.8) leads

$$(2.9)\qquad \frac{2^e-1}{\beta_1}\cdot\frac{2^e-1}{\beta_2}=\frac{q^f+1}{2}\cdot\frac{q^f-1}{q-1}=\frac{q^f+2^e}{\gamma_1}\cdot\frac{q^f-2^e}{\gamma_2},$$

where $\beta_1,\ \beta_2,\ \gamma_1$ and γ_2 are positive integers with $\beta_1\beta_2=q,\ \gamma_1\gamma_2=q-2$ and $(2^e+1)/\beta_1$, $(2^e-1)/\beta_2$, $(q^f+2^e)/\gamma_1$, $(q^f-2^e)/\gamma_2\in\mathbb{N}$. Let

$$A=\gcd\{(2^e+1)/\beta_1,\ (q^f+1)/2\},\qquad B=\gcd\{(2^e+1)/\beta_1,\ (q^f-1)/(q-1)\}$$

$$C=\gcd\{(2^e-1)/\beta_2,\ (q^f+1)/2\},\qquad D=\gcd\{(2^e-1)/\beta_2,\ (q^f-1)/(q-1)\}.$$

Then $A,\ B,\ C$ and D are odd. Furthermore they are relatively primes to each other.

Proposition 2.3 The following equations hold:

(a) $\begin{pmatrix} A \\ D \end{pmatrix}=\dfrac{1}{\gamma_2}\begin{pmatrix} q-1 & -\beta_2 \\ -\beta_1 & 2 \end{pmatrix}\begin{pmatrix} B \\ C \end{pmatrix}$;

(b) $\gamma_1=\beta_1A^2-\dfrac{q-1}{2}\beta_2D^2$;

(c) $\gamma_2 = \dfrac{q-1}{2}\beta_1 B^2 - \beta_2 C^2$.

Proof We have the following equations:

$$\begin{cases} 2^e+1=\beta_1 AB; & q^f+1=2AC; & q^f+2^e=\gamma_1 BC; \\ 2^e-1=\beta_2 CD; & q^f-1=(q-1)BD; & q^f-2^e=\gamma_2 AD. \end{cases}$$

Thus $\gamma_1 BC=(q^f+1)+(2^e-1)=2AC+\beta_2 CD$, and so $2A-\gamma_1 B+\beta_2 D=0$. Similarly, we have $P\boldsymbol{u}=\boldsymbol{o}$, where $P=\begin{pmatrix} 2 & -\gamma_1 & 0 & \beta_2 \\ \beta_1 & 0 & -\gamma_1 & q-1 \\ 0 & -\beta_1 & 2 & -\gamma_2 \\ -\gamma_2 & q-1 & -\beta_2 & 0 \end{pmatrix}$ and $\boldsymbol{u}=\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}$. Then $\mathrm{rank}(P)=2$ holds. Here $P\boldsymbol{u}=\boldsymbol{o}$ leads equation (a). Otherwise, we have $2\gamma_1 BC=\{(q^f+1)(2^e+1)-(q^f-1)(2^e-1)\}=2\beta_1 A^2BC-(q-1)\beta_2 BCD^2$, and so equation (b). Similarly, equation (c) is obtained.

□

Furthermore, from now on, we suppose that $q-2$ is a prime.

Proposition 2.4 We have the following results:

(a) If $q\equiv 5 \pmod{16}$ then $\beta_1=q,\ \beta_2=1,\ \gamma_1=q-2,\ \gamma_2=1$;

(b) If $q\equiv 13 \pmod{16}$ then $\beta_1=1,\ \beta_2=q,\ \gamma_1=q-2,\ \gamma_2=1$.

Proof If $q\equiv 5 \pmod{16}$ then $(q-1)/2\equiv 2 \pmod 8$. Therefore, if $\beta_1=1$ and $\beta_2=q$, from proposition 2.3(b), then $\gamma_1\equiv -1 \pmod 8$. This is a contradiction to $\gamma_1\equiv 1$ or $3 \pmod 8$. Thus $\beta_1=q$ and $\beta_2=1$ hold. Then, from proposition 2.3 (b), $\gamma_1\equiv 3 \pmod 8$ holds. Thus $\gamma_1=q-2$ and $\gamma_2=1$. Similarly, we can obtain result (b) from proposition 2.3(c).

□

Proposition 2.5 We have the following results:

(a) If $q\equiv 5 \pmod{16}$ with $q\neq 5$ then (2.7) has no solutions;

(b) If $q\equiv 13 \pmod{16}$ then (2.7) has no solutions.

Proof If a prime $q>5$ which $q-2$ is a prime, then $q\equiv 1 \pmod 3$ holds. Thus, from proposition 2.3 (c) and proposition 2.4, $C^2\equiv -1 \pmod 3$ holds. Therefore we have a contradiction.

□

Proposition 2.6 Let d be a square free integer. Let $\zeta=\alpha+\beta\sqrt{d}$ be an element of quadric field $\mathbb{Q}(\sqrt{d})$. Put $\zeta^n=\alpha_n+\beta_n\sqrt{d}$ for $n\in\mathbb{Z}$. Then both α_n and β_n are satisfied with the recurrence formula $X_{n+2}=2\alpha X_{n+1}-N(\zeta)X_n$.

Proof Since $\zeta^2=\alpha^2+\beta^2 d+2\alpha\beta\sqrt{d}=\alpha^2+\beta^2 d+2\alpha(\zeta-\alpha)$, it follows $\zeta^2=2\alpha\zeta-N(\zeta)$. Thus $\zeta^{n+2}=2\alpha\zeta^{n+1}-N(\zeta)\zeta^n$, and so $\alpha_{n+2}+\beta_{n+2}\sqrt{d}=2\alpha(\alpha_{n+1}+\beta_{n+1}\sqrt{d})-N(\zeta)(\alpha_n+\beta_n\sqrt{d})$. Therefore the proof is complete.

□

Theorem 2.2 We suppose that $q \equiv 5 \pmod 8$ and $q-2$ is a prime. Then equation (2.7) has a unique solution $(q,\ x,\ y) = (5,\ 5,\ 3)$.

Proof From proposition 2.5, we can show that (2.7) has no solutions except $q = 5$. Furthermore, it will be sufficient to prove this for the case of $q = 5$.

Then, from proposition 2.3 (c) and proposition 2.4 (a), we have $C^2 - 10B^2 = -1$. Let $\varepsilon = 3 + \sqrt{10}$ be a fundamental unit of $\mathbb{Q}(\sqrt{10})$. Put $\varepsilon^n = t_n + u_n\sqrt{10}$ for $n \in \mathbb{Z}$. Since $N(\varepsilon) = -1$, there exists $N \in \mathbb{N} \cup \{0\}$ such that $C = t_{2N+1}$ and $B = u_{2N+1}$. Furthermore, since $t_{2N+1} + u_{2N+1}\sqrt{10} = (t_{2N} + u_{2N}\sqrt{10})(3+\sqrt{10})$, we have $C = 3t_{2N} + 10u_{2N}$ and $B = t_{2N} + 3u_{2N}$. Thus, from proposition 2.3 (a), $A = t_{2N} + 2u_{2N}$ and $D = t_{2N} - 5u_{2N}$. Therefore, by substituting there results to $2AC = 5^f + 1 = 5^f + N(\varepsilon^{2N})$, we have

(2.10) $$5^f = 5t_{2N}^2 + 32t_{2N}u_{2N} + 50u_{2N}^2 .$$

Now, if $u_{2N} = 0$ then $N = 0$. Thus, since $t_0 = 1$ and $u_0 = 0$, it follows $f = 1$ and $(A,\ B,\ C,\ D) = (1,\ 1,\ 3,\ 1)$. Furthermore, $e = 1$ and $(q,\ x,\ y) = (5,\ 5,\ 3)$ are obtained by easy calculations. This is a solution of (2.7).

Suppose $u_{2N} > 0$. Hence $N \in \mathbb{N}$. Then, from (2.10), we have $t_{2N}u_{2N} \equiv 0$ (mod 5). Since $N(\varepsilon^{2N}) = 1$, $t_{2N} \not\equiv 0 \pmod 5$. Thus $u_{2N} \equiv 0 \pmod 5$ is obtained.

The equation (2.10) leads

(2.11) $$5^{f+1} = (5t_{2N} + 16u_{2N})^2 - 6u_{2N}^2 .$$

Let $\eta = 5 + 2\sqrt{6}$ be a fundamental unit of $\mathbb{Q}(\sqrt{6})$. Put $\eta^m = s_m + v_m\sqrt{6}$ for $m \in \mathbb{Z}$. And let $\xi = 1 + \sqrt{6}$ be an element of $\mathbb{Q}(\sqrt{6})$. Put $\xi^l = r_l + w_l\sqrt{6}$. We note that the class number of $\mathbb{Q}(\sqrt{6})$ has one. Then, since $N(\eta) = 1$ and $N(\xi) = -5$, there exists $M \in \mathbb{N} \cup \{0\}$ such that $(s_M + v_M\sqrt{6})(r_{2L} + w_{2L}\sqrt{6}) = (5t_{2N} + 16u_{2N}) + u_{2N}\sqrt{6}$ is satisfied, where $2L = f + 1$. Thus $\begin{pmatrix} s_M & 6v_M \\ v_M & s_M \end{pmatrix}\begin{pmatrix} r_{2L} \\ w_{2L} \end{pmatrix} = \begin{pmatrix} 5t_{2N} + 16u_{2N} \\ u_{2N} \end{pmatrix}$, and so

(2.12) $$\begin{pmatrix} r_{2L} \\ w_{2L} \end{pmatrix} = \frac{1}{N(\eta^M)}\begin{pmatrix} s_M & -6v_M \\ -v_M & s_M \end{pmatrix}\begin{pmatrix} 5t_{2N} + 16u_{2N} \\ u_{2N} \end{pmatrix} = \begin{pmatrix} 5s_M t_{2M} + (16s_M - 6v_M)u_{2M} \\ -5v_M t_{2M} + (s_M - 16v_M)u_{2M} \end{pmatrix}.$$

Since $u_{2N} \equiv 0 \pmod 5$, it follows $r_{2L} \equiv w_{2L} \equiv 0 \pmod 5$.

On the other hand, the sequence $\{r_l\}$ is satisfied with $r_0 = 1,\ r_1 = 1$ and $r_{l+2} = 2r_{l+1} + 5r_l$. That is $\cdots, 1,\ 1,\ 7,\ 19,\ 73,\ \cdots$. And, we have $r_{l+2} \equiv 2r_{l+1} \pmod 5$ for $l \in \mathbb{N} \cup \{0\}$. Thus $r_l \not\equiv 0 \pmod 5$ holds for $l \in \mathbb{N}$. This is a contradiction to $r_{2L} \equiv 0 \pmod 5$. Therefore, if $u_{2N} > 0$, then (2.7) has no solutions.

□

3. Equation $a^{x_1}-b^{y_1}=a^{x_2}-b^{y_2}=1$

Let a and b be positive integers with $a,\ b>1$. Let $x_1,\ x_2,\ y_1$ and y_2 be integers with $x_1<x_2$ and $y_1<y_2$. In this section, we consider the equation

(3.1) $\quad a^{x_1}-b^{y_1}=a^{x_2}-b^{y_2}=1$.

The equation (3.1) leads

(3.2) $$\frac{a^{x_{12}}-1}{b^{y_1}}=\frac{b^{y_{12}}-1}{a^{x_1}}\in\mathbb{N},$$

where $x_{12}=x_2-x_1$ and $y_{12}=y_2-y_1$.

Furthermore, since $a^{x_1}-b^{y_1}=1$, (3.2) leads

(3.3) $$\frac{a^{x_{12}}-1}{a^{x_1}-1}=\frac{b^{y_{12}}-1}{b^{y_1}+1}\in\mathbb{N}.$$

Now, let $s=[x_{12}/x_1]$ and $t=[y_{12}/y_1]$, where $[\ \cdot\ ]$ is Gauss' notation. Then we have

(3.4) $$\frac{a^{x_{12}}-1}{a^{x_1}-1}=a^{x_{12}-x_1}+\cdots+a^{x_{12}-sx_1}+\frac{a^{x_{12}-sx_1}-1}{a^{x_1}-1}$$

and

(3.5) $$\frac{b^{y_{12}}-1}{b^{y_1}+1}=b^{y_{12}-y_1}-\cdots+(-1)^{t-1}b^{y_{12}-ty_1}+\frac{(-1)^t b^{y_{12}-tx_1}-1}{b^{y_1}+1}.$$

Therefore, from (3.3), $x_1\,|\,x_{12}$, $y_1\,|\,y_{12}$ and $t\equiv 0\ (\mathrm{mod}\ 2)$ hold. Thus there exist $e,\ f\in\mathbb{N}$ such that $x_{12}=ex_1$ and $y_{12}=2fy_1$ are satisfied. Let $X=a^{x_1}$ and $Y=b^{y_1}$. Then we have

(3.6) $\quad X-Y=1$.

And (3.3) leads

(3.6) $$\frac{X^e-1}{X-1}=\frac{Y^{2f}-1}{Y+1}.$$

Furthermore, (3.6) and (3.7) lead

(3.8) $\quad (Y+1)^{e+1}=X^{e+1}=Y^{2f+1}+1=(X-1)^{2f+1}+1$.

Since $X^{e+1}=(X-1)^{2f+1}+1=\sum_{k=1}^{2f+1}(-1)^{k-1}{}_{2f+1}C_kX^k$, we have $(2f+1)X\equiv 0\ (\mathrm{mod}\ X^2)$, and so $2f+1\equiv 0\ (\mathrm{mod}\ X)$. Thus there exists $f_1\in\mathbb{N}$ such that $2f+1=Xf_1$ is satisfied. Then, since $X\equiv 1\ (\mathrm{mod}\ 2)$ holds, we can suppose $X\geq 3$. And $X\equiv 1\ (\mathrm{mod}\ 2)$ leads $Y\equiv 0\ (\mathrm{mod}\ 2)$.

Similarly, since $Y^{Xf_1}=(Y+1)^{e+1}-1=\sum_{k=1}^{e+1}{}_{e+1}C_kY^k$, we have $(e+1)Y\equiv 0\ (\mathrm{mod}\ Y^2)$, and so $e+1\equiv 0\ (\mathrm{mod}\ Y)$. Thus there exists $e_1\in\mathbb{N}$ such that $e+1=Ye_1$ is satisfied.

Here, if $Ye_1=2$ then $Y=2$ and $e_1=1$. Therefore, by easy calculations, we can obtain a solution $(a,\ b,\ x_1,\ x_2,\ y_1,\ y_2)=(3,\ 2,\ 1,\ 2,\ 1,\ 3)$ of (3.1). That is $3^1-2^1=3^2-2^3=1$.

From now on, we suppose $Ye_1\geq 4$. Then (3.8) leads

(3.9) $\quad (Y+1)^{Ye_1}=X^{Ye_1}=Y^{Xf_1}+1=(X-1)^{Xf_1}+1$.

If $Y=2$ then $e_1 \geq 2$. And, from (3.6) and (3.9), $3^{2e_1}-2^{3f_1}=1$ holds. This is a contradiction to theorem 1.1 in paper (5). Thus we can suppose $Y \geq 4$.

Since $X^{Ye_1}=(X-1)^{Xf_1}+1=X^2 f_1 - X^3 f_1\{(Xf_1-1)/2\}+\sum_{k=3}^{Xf_1}(-1)^{k-1}{}_{Xf_1}C_k X^k$, we have $X^2 f_1 \equiv 0$ (mod X^3), and so $f_1 \equiv 0$ (mod X). Thus there exists $f_2 \in \mathbb{N}$ such that $f_1 = Xf_2$ is satisfied.

Similarly, since $Y^{X^2 f_2}=(Y+1)^{Ye_1}-1=Y^2 e_1+(Y/2)Y^2 e_1(Ye_1-1)+\sum_{k=3}^{Ye_1}{}_{Ye_1}C_k Y^k$, we have $Y^2 e_1 \equiv 0$ (mod $(Y/2)Y^2$), and so $e_1 \equiv 0$ (mod $Y/2$). Thus there exists $e_2 \in \mathbb{N}$ such that $e_1=(Y/2)e_2$ is satisfied.

Then (3.9) leads

(3.10) $$(Y+1)^{\left(\frac{Y}{2}\right)Ye_2}=X^{\left(\frac{Y}{2}\right)Ye_2}=Y^{X^2 f_2}+1=(X-1)^{X^2 f_2}+1.$$

Proposition 3.1 Let a, n, k and Z be positive integers with $Z>1$. Then

(3.11) $$\left(\frac{aZ^{k-2}}{k}\right)\cdot {}_{aZ^n-1}C_{k-1} \in \mathbb{N}$$

hold for $3 \leq k \leq aZ^n$

Proof We have

$$ {}_{aZ^n}C_k Z^k=\left\{\left(aZ^n/k\right)\cdot {}_{aZ^n-1}C_{k-1}\right\}Z^k=\left\{\left(aZ^{k-2}/k\right)\cdot {}_{aZ^n-1}C_{k-1}\right\}Z^{n+2}\in\mathbb{N}.$$

Therefore, any prime factor of k which is not a prime factor of Z is a prime factor of $a\cdot {}_{aX^{n-1}}C_{k-1}$. And the number of prime factors in k is $k-2$ or less. Thus (3.11) is satisfied.

□

Proposition 3.2 Let n be a positive integer with $n \geq 2$. Then there exist sequences $\{e_n\}$ and $\{f_n\}$ such that

(3.12) $$(Y+1)^{\left(\frac{Y}{2}\right)^{n-1}Ye_n}=X^{\left(\frac{Y}{2}\right)^{n-1}Ye_n}=Y^{X^n f_n}+1=(X-1)^{X^n f_n}+1$$

is satisfied.

Proof The proof is by induction on n. The equation (3.12) is already proven for $n=2$. We suppose that (3.12) holds for $n=k$. Then, since

$$\begin{aligned} X^{\left(\frac{Y}{2}\right)^{k-1}Ye_k} &= (X-1)^{X^k f_k}+1 = X^{k+1}f_k - X^{k+2}f_k\left(\frac{X^k f_k-1}{2}\right)+\sum_{j=3}^{X^k f_k}(-1)^{j-1}{}_{X^k f_k}C_j X^j \\ &= X^{k+1}f_k - X^{k+2}f_k\left(\frac{X^k f_k-1}{2}\right)+X^{k+2}\sum_{j=3}^{X^k f_k}(-1)^{j-1}\left\{\left(\frac{X^{j-2}f_k}{j}\right){}_{X^k f_k-1}C_{j-1}\right\} \end{aligned}$$

holds, we have $X^{k+1}f_k \equiv 0$ (mod X^{k+2}), and so $f_k \equiv 0$ (mod X). Thus there exists $f_{k+1}\in\mathbb{N}$ such that $f_k = Xf_{k+1}$ is satisfied. Similarly, since

$$\begin{aligned} Y^{X^k f_k} &= (Y+1)^{\left(\frac{Y}{2}\right)^{k-1}Ye_k}-1=\left(\frac{Y}{2}\right)^{k-1}Y^2 e_k+\left(\frac{Y}{2}\right)^{k}Y^2 e_k\left\{\left(\frac{Y}{2}\right)^{k-1}Ye_k-1\right\}+\sum_{j=3}^{\left(\frac{Y}{2}\right)^{k-1}Ye_k}{}_{\left(\frac{Y}{2}\right)^{k-1}Ye_k}C_j Y^j \\ &= \left(\frac{Y}{2}\right)^{k-1}Y^2 e_k+\left(\frac{Y}{2}\right)^{k}Y^2 e_k\left\{\left(\frac{Y}{2}\right)^{k-1}Ye_k-1\right\}+\left(\frac{Y}{2}\right)^{k-1}Y^3\sum_{j=3}^{\left(\frac{Y}{2}\right)^{k-1}Ye_k}\left\{\left(\frac{Y^{j-2}e_k}{j}\right){}_{\left(\frac{Y}{2}\right)^{k-1}Ye_k-1}C_{j-1}\right\} \end{aligned}$$

holds, we have $(Y/2)^{k-2}Y^2 e_k \equiv 0$ (mod $(Y/2)^k Y^2$), and so $e_k \equiv 0$ (mod $Y/2$). Thus there exists $e_{k+1}\in\mathbb{N}$ such that

$e_k = Ye_{k+1}$ is satisfied. Therefore (3.12) holds for $n = k+1$.

□

If $Ye_1 \geq 4$ is supposed, by applying proposition 3.2, we can show that (3.8) has no solutions. Thus the following theorem is satisfied.

Theorem 3 The equation (3.1) has a unique solution $(a, b, x_1, x_2, y_1, y_2) = (3, 2, 1, 2, 1, 3)$.

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References

(1) P.Mihăilescu, "Primary Cyclotomic Units and a Proof of Catalan's Conjecture", J. Reinen Angew. Math. 572, pp.167-pp.195(2004).
(2) M.A.Bennett, "On Some Exponential Equation of S.S.Pillai", Canad. J. Math. 53(5), pp.897-pp.922(1980).
(3) R.Balasubramanian and T.N.Shorey, "On the Equation $a(x^m-1)/(x-1) = b(y^n-1)/(y-1)$", Math. Scand. 46, pp.177-pp.182(1980).
(4) T.N.Shorey, "On the Equation $a(x^m-1)/(x-1) = b(y^n-1)/(y-1)$ (II)", Hardy-Ramanujan J. 7, pp.1-pp.10(1984).
(5) Y.Motoda, "On the Equation $(p^{2e+1}+1)/(p+1) = (q^3+1)/(q+1)$", Res. Rep. Yatsushiro Nat. Coll. Tech. 13, pp.89-pp.92(1990).