Paper

Rees property and its related properties of modules

Satoru Isogawa*

Abstract Properties of ideals of a commutative Noetherian local ring or a Noetherian standerd graded commutative algebra over a field, called the Rees property, the second Rees property, the m-fullness and the fullness, can naturally be extended to the properties of modules. We introduce the property of modules called "weak" m-fullness and "weak" fullness. We show that the Rees property and the m-fullness are equivalent in the class of modules with weak m-fullness, and also that the second Rees property and the fullness are equivalent in the class of modules with weak fullness.

Keywords : Rees property, Second Rees property, fullness, m-fullness, weak Lefschetz property.

1. Introduction

Let R be a Noetherian standard graded commutative algebra over a field k, with its irrelevant maximal ideal $\mathfrak{m}$, i.e., $R=\bigoplus_{i=0}^{\infty}R_i$ with $R_0=k$, $R=k[R_1]$ and $\mathfrak{m}=\bigoplus_{i=1}^{\infty}R_i$. For any finitely generated graded R-module M. We denote the length of M by $l(M)$, the minimal number of generators of M by $\mu(M)\left(=l(M/\mathfrak{m}M)\right)$ and put $\mathrm{sub}(M):=\left\{N \middle| \text{a graded submodule } N\subseteq M \text{ with } l(M/N)<\infty\right\}$. For any $N\in\mathrm{sub}(M)$, we denote the type of N in M by $\tau_M(N)=\tau(N)\left(=l\left(N\underset{M}{:}\mathfrak{m}/N\right)\right)$ and the *t-value* of N in M (Definition 3.3 in [5]) by $t_M(N)=t(N)\left(=\min_{l\in R_1}t_l(N)\right)$, where $N\underset{M}{:}S=N:S:=\left\{x\in M \middle| Sx\subseteq N\right\}$ for a subset S of R and $t_l(N)=l\left(N\underset{M}{:}l/N\right)$. Further we denote:

$$\underline{\mu_M}(N)=\underline{\mu}(N):=\max\left\{\mu(N')\middle|N'\supseteq N, N'\in\mathrm{sub}(M)\right\},$$

$$\underline{\tau_M}(N)=\underline{\tau}(N):=\max\left\{\tau(N')\middle|N'\supseteq N, N'\in\mathrm{sub}(M)\right\}.$$

(1) A homogeneous ideal I of R is called $\mathfrak{m}$*-full* if $\mathfrak{m}I:l=I$ for some $l\in R_1$, i.e., $\mu(I)=t(\mathfrak{m}I)$ ([9]).

(2) A homogeneous ideal I of R is called *full* if $I:\mathfrak{m}=I:l$ for some $l\in R_1$, i.e., $\tau(I)=t(I)$ ([2]).

(3) A homogeneous ideal I of R has the *Rees property* if if $\mu(J)\le\mu(I)$ for all homogeneous ideal $J\supseteq I$, i.e. $\mu(I)=\underline{\mu}(I)$ ([9]).

(4) A homogeneous ideal I of R has the second *Rees property* if $\tau(J)\le\tau(I)$ for all homogeneous ideal $J\supseteq I$, i.e. $\tau(I)=\underline{\tau}(I)$.

The notion of the second Rees property has been introduced by J. Watanabe in a private conversation with T. Harima in 2008.

These properties of ideals, which have been studied by many authors (e.g., [1]- [4], [7]-[11]), are naturally extend to the properties of submodules of a given module as follows:

Definition 1. *Let M be a finitely generated graded R-module.*

(1) $\mathfrak{m}\text{-Full}(M):=\{\textit{The set of } \mathfrak{m}\textit{-full (graded) submodules of } M\}=\left\{N\in\mathrm{sub}(M)\middle|\mu(N)=t(\mathfrak{m}N)\right\}$.

(2) $\mathrm{Full}(M):=\{\textit{The set of full (graded) submodules of } M\}=\left\{N\in\mathrm{sub}(M)\middle|\tau(N)=t(N)\right\}$.

(3) $\mathrm{Rees}(M):=\{\textit{The set of Rees (graded) submodules of } M\}=\left\{N\in\mathrm{sub}(M)\middle|\mu(N)=\underline{\mu}(N)\right\}$.

(4) $\mathrm{SRees}(M):=\{\textit{The set of second Rees (graded) submodules of } M\}=\left\{N\in\mathrm{sub}(M)\middle|\tau(N)=\underline{\tau}(N)\right\}$.

We have already studied these properties in purely combinatorial way in [6]. There have been intorduced the notions "weak" m-fullness, "weak" fullness, "restricted" second Rees property and "restricted" fullness in [6]. In the case of modules,

* Faculty of Liberal Studies
2627 Hirayama-shinmachi Yatsushiro-shi Kumamoto, Japan 866-8501
This work was supported by Grants-in-Aid for Scientific Research (C) 23540052 of JSPS.

these notions can be sated as follows:

Definition 2. *Let* M *be a finitely generated graded* R *-module.*

(1) $\text{w-}\mathfrak{m}\text{-Full}(M) := \{\textit{The set of } \mathfrak{m}\textit{-full } (\textit{graded}) \textit{ submodules of } M\} = \{N \in \text{sub}(M) | \underline{\mu}(N) \ge t(\mathfrak{m}N)\}$.

(2) $\text{w-Full}(M) := \{\textit{The set of full } (\textit{graded}) \textit{ submodules of } M\} = \{N \in \text{sub}(M) | \underline{\tau}(N) \ge t(N)\}$.

(3) $\text{Full}'(M) := \{\textit{The set of restrictedly full } (\textit{graded}) \textit{ submodules of } M\} = \{N \in \text{sub}(M) | \mu(N:\mathfrak{m}) = \tau(N) = t(N)\}$.

(4) $\text{SRees}'(M) := \{\textit{The set of restricted second Rees } (\textit{graded}) \textit{ submodules of } M\} = \{N \in \text{sub}(M) | \mu(N:\mathfrak{m}) = \tau(N) = \underline{\tau}(N)\}$.

Let $(\ )^{\vee} := \text{Hom}_k(\ \ ,k)$ be the k -dual operator. We define a inclusion-reversing operator $(\ )^{\circ} : \text{sub}(M) \to \text{sub}(M^{\vee})$ as follows:

Definition 3. *Let* M *be a finitely generated graded* R *-module with* $l(M) < \infty$ *. For any* $N \in \text{sub}(M)$ *,*

$$N^{\circ} := (M/N)^{\vee} \in \text{sub}(M^{\vee})$$

where we assume that $(M/N)^{\vee} \subseteq M^{\vee}$ *via* $(M/N)^{\vee} \xrightarrow{\pi^{\vee}} M^{\vee}$ *: the* k *-dual of the natural projection* $\pi : M \to M/N$ *.*

On the other hand, a homogeneous ideal $I \subseteq R$ is said "having the weak Lefschetz property" if there exists $l \in R_1$ such that each map $\times l : (R/I)_i \to (R/I)_{i+1}$, defined by multiplication by l , is surjective or injective for any integer i where $(R/I)_i$ is the i -th graded compnent of R/I . This property of ideals have also been studied by many authors, for example, consult the listed papers in the reference of [4]. If a homogeneous ideal $I \subseteq R$ with $l(R/I) < \infty$ has "the weak Lefschetz property", then it is well known that $h_{R/I}$ ($h_{R/I}(i) := \dim_k (R/I)_i$), the Hilbert function of R/I , is unimodul, i.e., $h_{R/I}(0) \le \cdots \le h_{R/I}(j) \ge h_{R/I}(j+1) \ge \cdots$ for some integer $j \ge 0$. We extend this property to modules as follows:

Definition 4. *Let* M *be a finitely generated graded* R *-module. For any* $N \in \text{sub}(M)$ *, we call that* N *has "restricted weak Lefschetz property w.r.t.* M *" if there exists* $l \in R_1$ *such that each map* $\times l : (M/N)_i \to (M/N)_{i+1}$ *, defined by multiplication by* l *, is surjective or injective for any integer* i *where* $(M/N)_i$ *is the i-th graded compnent of* M/N *, and further if* $h_{M/N}$ *(* $h_{M/N}(i) := \dim_k (M/N)_i$ *), the Hilbert function of* M/N *, is unimodal. In this case,* l *is called a "Lefschetz element of* N *w.r.t* M *".*

Throughout this paper, we study the Rees property and its related properties of modules by applying the results of [6] to modules. In section 2, we describe the relations among the Rees property and its related properties. In section 3, we show that the existance of the m-full closure and the full closure of a module. In section 4, in terms of restricted weak Lefschetz property, we describe the condition under which the second Rees property and the fullness are equivalent, and also describe the condition under which the second Rees property and the fullness are equivalent. In section5, We study how the Rees property and its related properties behave under taking the inclusion-reversing bijection $(\ )^{\circ} : \text{sub}(M) \to \text{sub}(M^{\vee})$.

2. Relations among Rees property and its related properties of modules

In what follows, we assume that M is a finitely generated graded R -module. We introduce a partial ordere in $\text{sub}(M)$ by inclusion-reverse order, i.e., for any $N, N' \in \text{sub}(M)$, we define $N \ge N'$ if and only if $N \subseteq N'$. We fix some notations as follows:

Notation1. *For any* $N, N' \in \text{sub}(M)$ *,* $l \in R_1$ *and* $\Omega \subseteq \text{sub}(M)$ *:*

(1) $N^{\#} := \mathfrak{m}N,\ N^{l} := lN, N_{\#} := N:\mathfrak{m}, N_{l} := N:l \textit{ and } rN := l(M/n)$.

(2) $r^{\#}N := rN^{\#} - rN = \mu(N),\ r^{l}N := rN^{l} - rN, r_{\#}N := rN - rN_{\#} = \tau(N), \textit{ and } r_{l}N := rN - rN_{l} = t_{l}(N)$.

(3) $\underline{r^{\#}}N := \max\{r^{\#}L = \mu(L) | L \in \text{sub}(M) \textit{ with } L \le N\}, \underline{r_{\#}}N := \max\{r_{\#}L = \tau(L) | L \in \text{sub}(M) \textit{ with } L \le N\}_{\#}$.

(4) $N \wedge N' := N + N'$.

It is easy to check that the following hold (see [6]):

・ $\text{sub}(M)$ is a ranked poset with rank function r .

・ $N \wedge N'$ is the infimum of N and N' .

Putting $\mathcal{P} = \text{sub}(M)$ and $\Phi = R_1$ in Definition 2 and Definition 3 in [6], the following hold:

- $\mathfrak{m}\text{-Full}_\Phi\mathcal{P}=\mathfrak{m}\text{-Full}(M)$, $\text{Full}_\Phi\mathcal{P}=\text{Full}(M)$, $\text{Rees}\mathcal{P}=\text{Rees}(M)$, $\text{SRees}\mathcal{P}=\text{SRees}(M)$.
- $\text{w-}\mathfrak{m}\text{-Full}_\Phi\mathcal{P}=\text{w-}\mathfrak{m}\text{-Full}(M)$, $\text{w-Full}_\Phi\mathcal{P}=\text{w-Full}(M)$.

Lemma1. *For* $\mathcal{P}=\text{sub}(M)$*, the following hold:*

(1) *For any* $N\in\mathcal{P}$ *and* $l\in R_1$, $N_l\le(N^l)_l\le N\le(N_l)^l\le N^l$ *and* $N_l\le N_\#\le N^\#\le N^l$.

(2) *For any* $N\in\mathcal{P}$ *and* $l\in R_1$, $r_lN=r(N\wedge L^l)-rL$ *whenever* $L\le N_l$.

(3) *For any* $N\in\mathcal{P}$ *and* $l_1,l_2\in R_1\cup\{\#\}$, $\left(N_{l_1}\right)_{l_2}=\left(N_{l_2}\right)_{l_1}$ *and* $\left(N^{l_1}\right)^{l_2}=\left(N^{l_2}\right)^{l_1}$.

Proof. (1) and (3) directly follow from the definitions.

(2) Using the following exact sequence:

$$0\to\frac{N:l}{N}\to\frac{L}{N}\xrightarrow{\times l}\frac{L}{N}\to\frac{L}{N+lL}\to 0\ (\text{exact}),$$

we have $r_lN=l\left(\frac{N:l}{N}\right)=l\left(\frac{L}{N+lL}\right)=r(N\wedge L^l)-rL$. □

From Lemma 1, $\mathcal{P}=\text{sub}(M)$ satisfies Condition 3(and Condition 1) in [6]. Therefore we can apply the results in [6] to $\mathcal{P}=\text{sub}(M)$.

Definition 5. *For any finitely generated graded* R*-module* M*, we define*

$$d_\mu(M):=\max\{\mu(N)\mid N\in\text{sub}(M)\}\ \textit{(if it exists, otherwise } d(M):=\infty\textit{)},$$

$$d_\tau(M):=\max\{\tau(N)\mid N\in\text{sub}(M)\}\ \textit{(if it exists, otherwise } d'(\mathcal{P}):=\infty\textit{)}.$$

Proposition 1. $d_\mu(M)=d_\tau(M)$.

Proof. This follows from Proposition 1 in [6].

We put $\text{sub}(M)^\#:=\{N^\#=\mathfrak{m}N\mid N\in\text{sub}(M)\}$, $\text{sub}(M)_\#:=\{N_\#=N:\mathfrak{m}\mid N\in\text{sub}(M)\}$.

Lemma 2. *For any* $N,N'\in\text{sub}(M)$ *and* $l\in R_1$*, the following hold:*

(1) $\mu(N)\le\tau(\mathfrak{m}N)=\mu(\mathfrak{m}N:\mathfrak{m})$. *Especially* $\mu(N)=\tau(\mathfrak{m}N)$ *if and only if* $\mathfrak{m}N:\mathfrak{m}=N$.

(2) $\tau(N)\le\mu(N:\mathfrak{m})=\tau(\mathfrak{m}(N:\mathfrak{m}))$. *Especially* $\tau(N)=\mu(N:\mathfrak{m})$ *if and only if* $\mathfrak{m}(N:\mathfrak{m})=N$.

(3) $\underline{\mu}(N)=\underline{\tau}(\mathfrak{m}N)$.

(4) $\underline{\tau}(N)=\underline{\mu}(N:\mathfrak{m})$ *if* $\mathfrak{m}(N:\mathfrak{m})=N$.

(5) $\tau(N)\le t(N)\le t_l(N)$. *Especially* $\tau(N)=t_l(N)(=t(N))$ *if and only if* $N:l=N:\mathfrak{m}$.

(6) $\mu(N)\le t(\mathfrak{m}N)\le t_l(\mathfrak{m}N)$. *Especially* $\mu(N)=t_l(\mathfrak{m}N)(=t(\mathfrak{m}N))$ *if and only if* $\mathfrak{m}N:l(=\mathfrak{m}N:\mathfrak{m})=N$.

(7) *If* $\mathfrak{m}N:l\subseteq N'$*, then* $t_l(\mathfrak{m}N:l)\ge\mu(N')$. *Especially* $\underline{\mu}(\mathfrak{m}N:l)\le t_l(\mathfrak{m}N)$.

(8) *If* $N\subseteq N'$*, then* $t_l(N)\ge t_l(N')(\ge\tau(N'))$. *Especially* $\underline{\tau}(N)\le t(N)<\infty$.

(9) $\underline{\mu}(N)\le t(\mathfrak{m}N)<\infty$.

(10) $\text{sub}(M)^\#=\{N\in\text{sub}(M)\mid\tau(N)=\mu(N:\mathfrak{m})\}$, $\text{sub}(M)_\#=\{N\in\text{sub}(M)\mid\mu(N)=\tau(\mathfrak{m}N)\}$.

Proof. (1)-(4) : These follow from Lemma 1 (3)-(6) in [6].

(5),(6) : These follow from Lemma 3 (1)-(4) in [6].

(7)-(9) : These follow from Lemma 6 in [6].

(10): This follows from (1), (2) and Lemma 1 (1) in [6]. □

Remark 1. *From Lemma 2(8),(9) (or Remark 5 in [6]), we have*

$$\text{w-Full}(M)=\{N\in\text{sub}(M)\mid\underline{\tau}(N)(=\underline{\mu}(N:\mathfrak{m}))=t(N)\},$$

$$\text{w-}\mathfrak{m}\text{-Full}(M)=\{N\in\text{sub}(M)\mid\underline{\mu}(N)(=\underline{\tau}(\mathfrak{m}N))=t(\mathfrak{m}N)\}.$$

Lemma 3. $\mathfrak{m}N\in\text{w-Full}(M)$ *if and only if* $N\in\text{w-}\mathfrak{m}\text{-Full}(M)$.

Proof. This follows from Remark 1 and Lemma 2 (3) . □

Lemma 4. *For any* $N\in\text{sub}(M)$*, the following are equivalent:*

a) $N\in\text{w-Full}(M)$*, i.e.,* $\underline{\tau}(N)=t(N)$.

b) *There exists* $N'\in\text{sub}(M)$ *with* $N'\supseteq N$ *such that* $\tau(N')=t(N)$.

c) *There exists* $N' \in \mathrm{sub}(M)$ *with* $N' \supseteq N$ *and* $l \in R_1$ *such that* $\tau(N') \geq t_l(N)$.

Proof. a)$\Rightarrow$b) : Take $N' \supseteq N$ such that $\underline{\tau}(N) = \tau(N')$, then $\underline{\tau}(N') = t(N)$ by our assumption.

b)$\Rightarrow$c): Take $l \in R_1$ such that $t(N) = t_l(N)$, then $\tau(N') \geq t_l(N)$ by our assumption.

c)$\Rightarrow$a) : From Lemma 2(8), we have $\underline{\tau}(N) \leq t(N)$. On the other hand, $\underline{\tau}(N) \geq \tau(N') \geq t_l(N) \geq t(N)$ holds by our assumption. This implies $\underline{\tau}(N) = t(N)$. □

From Lemma 2(10), putting $\mathcal{P} = \mathrm{sub}(M)$ and $\Phi = R_1$ in Definition 2 and Definition 3 in [6], the following hold:

$$\mathrm{SRees}'\mathcal{P} = \mathrm{SRees}(M),\ \mathrm{Full}'_{\Phi}\mathcal{P} = \mathrm{Full}'(M).$$

We put

$$\text{w-Full}'(M) := \left\{ N \in \mathrm{sub}(M) \middle| \underline{\tau}(N) = t(N), \tau(N) = \mu(N : \mathfrak{m}) \right\},$$

$$\text{w-}\mathfrak{m}\text{-Full}^{\dagger}(M) := \left\{ N \in \mathrm{sub}(M) \middle| \underline{\mu}(N) \geq t(\mathfrak{m}N),\ \mu(N) = \tau(\mathfrak{m}N) \right\}.$$

Theorem 1. *There are one-to-one correspondences:*

$$(1)\quad \mathrm{SRees}(M) \underset{:\mathfrak{m}}{\overset{\times\mathfrak{m}}{\rightleftarrows}} \mathrm{SRees}'(M),$$

$$(2)\ \mathfrak{m}\text{-Full}(M) \underset{:\mathfrak{m}}{\overset{\times\mathfrak{m}}{\rightleftarrows}} \mathrm{Full}'(M),$$

$$(3)\ \text{w-}\mathfrak{m}\text{-Full}^{\dagger}(M) \underset{:\mathfrak{m}}{\overset{\times\mathfrak{m}}{\rightleftarrows}} \text{w-Full}'(M).$$

Proof. This follows from Theorem 1 and 2 in [6]. □

Theorem 2. *The following hold:*

(1) $\mathfrak{m}\text{-Full}(M) \subseteq \mathrm{Rees}(M)$, $\mathrm{Full}(M) \subseteq \mathrm{SRees}(M)$, $\mathfrak{m}\text{-Full}(M) \subseteq \mathrm{Full}(M)$.

(2) *If* $N \in \text{w-}\mathfrak{m}\text{-Full}(M)$*, then the following two conditions are equivalent:*

a) $N \in \mathrm{Rees}(M)$.

b) $N \in \mathfrak{m}\text{-Full}(M)$.

(3) *If* $N \in \text{w-Full}(M)$*, then the following two conditions are equivalent:*

a) $N \in \mathrm{SRees}(M)$.

b) $N \in \mathrm{Full}(M)$.

(4) $N \in \mathrm{Rees}(M) \setminus \mathfrak{m}\text{-Full}(M)$ *if and only if* $N \in \mathrm{Rees}(M)$ *and* $\underline{\mu}(N) < t(\mathfrak{m}N)$.

(5) $N \in \mathrm{SRees}(M) \setminus \mathrm{Full}(M)$ *if and only if* $N \in \mathrm{SRees}(M)$ *and* $\underline{\tau}(N) < t(N)$.

Proof. This follows from Theorem 3 and Corollary 1 in [6]. □

3. The m-full closure and the full closure of modules

3.1 Preliminary

Let W be a finte dimensional k-vector space with a fixed tuple of base $\underline{e} = (e_1, \cdots, e_m)$. For any k-linear endomorphism $\varphi \in \mathrm{Hom}_k(W, W)$, we denote its matrix representation w.r.t. $\underline{e}$ by $\mathrm{Mat}(\varphi)$. Let $\mathbb{A}^n(k)$ be the affine n-space over k with its coordinate ring $k[x_1, \cdots, x_n]$. We denote $V(J) := \left\{ (a_1, \cdots, a_n) \in \mathbb{A}^n(k) \middle| f(a_1, \cdots, a_n) = 0 \text{ for all } f \in J \right\}$ the Zariski closed set in $\mathbb{A}^n(k)$ defined by an ideal $J \subseteq k[x_1, \cdots, x_n]$ and $D(J) := \mathbb{A}^n(k) \setminus V(J)$ the Zariski open set defined by an ideal J. For a finite set of k-linear endomorphisms $\{\varphi_1, \cdots, \varphi_n\} \subseteq \mathrm{Hom}_k(V, V)$ we put:

$$U_{\mathrm{rank} \geq i}\left(\varphi_1, \cdots, \varphi_n; \mathbb{A}^n(k)\right) := \left\{ (a_1, \cdots, a_n) \in \mathbb{A}^n(k) \middle| \mathrm{rank}\left(\mathrm{Mat}\left(a_1\varphi_1 + \cdots + a_n\varphi_n\right)\right) \geq i \right\},$$

$$V_{\mathrm{rank} \leq i}\left(\varphi_1, \cdots, \varphi_n; \mathbb{A}^n(k)\right) := \left\{ (a_1, \cdots, a_n) \in \mathbb{A}^n(k) \middle| \mathrm{rank}\left(\mathrm{Mat}\left(a_1\varphi_1 + \cdots + a_n\varphi_n\right)\right) \leq i \right\}.$$

Lemma 5. *For any non-negative integer i, there exists a homogeneous ideal* $J \subseteq (x_1, \cdots, x_n)$ *such that* $V_{\mathrm{rank} \leq i}\left(\varphi_1, \cdots, \varphi_n; \mathbb{A}^n(k)\right) = V(J)$*. Especially,* $U_{\mathrm{rank} \geq i}\left(\varphi_1, \cdots, \varphi_n; \mathbb{A}^n(k)\right)$ *is a Zariski open set in* $\mathbb{A}^n(k)$.

Proof. If we put $J = ($ generated by all $i \times i$ minors of the Matrix $\mathrm{Mat}\left(x_1\varphi_1 + \cdots + x_n\varphi_n\right)$), then J is a homogeneous ideal, $J \subseteq (x_1, \cdots, x_n)$ and $V_{\mathrm{rank} \leq i}\left(\varphi_1, \cdots, \varphi_n; \mathbb{A}^n(k)\right) = V(J)$. □

3.2 Existance of the m-full closure and the full closure

Fixing a base $\{y_1,\cdots,y_n\}$ of R_1, we identify R_1 as $\mathbb{A}^n(k)$, by $R_1 \ni a_1y_1+\cdots+a_ny_n \leftrightarrow (a_1,\cdots,a_n)\in\mathbb{A}^n(k)$. For $N\in \mathrm{sub}(M)$, we put $D(N):=\{l\in R_1 | t_l(N)=t(N)\}\subseteq R_1\simeq\mathbb{A}^n(k)$.

Lemma 6. *The following hold:*

(1) $D(N)$ *is a Zariski open set in* $R_1\simeq\mathbb{A}^n(k)$ *for any* $N\in\mathrm{sub}(M)$.

(2) *If* $N\in\mathfrak{m}\text{-Full}(M)$, *then* $D(\mathfrak{m}N)=\{l\in R_1 | \mathfrak{m}N:l=N\}\neq\emptyset$.

(3) *If* $N\in\mathrm{Full}(M)$, *then* $D(N)=\{l\in R_1 | N:l=N:\mathfrak{m}\}\neq\emptyset$.

(4) *If* $N,L\in\mathfrak{m}\text{-Full}(M)$, *then* $N\cap L\in\mathfrak{m}\text{-Full}(M)$.

(5) *If* $N,L\in\mathrm{Full}(M)$, *then* $N\cap L\in\mathrm{Full}(M)$.

Proof. (1) :Put $\varphi_i=\times y_i : M/N \xrightarrow{\times y_i} M/N$ the k-linear endomorphism on M/N defined by taking multiplication by $y_i\ (1\le i\le n)$. We have

$$U_{\mathrm{rank}\ge n-t(n)}(\varphi_1,\cdots,\varphi_n;\mathbb{A}^n(k))\simeq\{l=a_1y_1+\cdots+a_ny_n\in R_1\simeq\mathbb{A}^n(k) | \dim_k\ker(a_1\varphi_1+\cdots+a_n\varphi_n)=t_l(n)\le t(n)\}=D(N).$$

By Lemma 5, $D(N)$ is a Zariski open set in $R_1\simeq\mathbb{A}^n(k)$.

(2), (3): These follow from Lemma 2(5), (6).

(4): From (1) and (2), $D(\mathfrak{m}N)\cap D(\mathfrak{m}L)\neq\emptyset$. So take an element $l\in D(\mathfrak{m}N)\cap D(\mathfrak{m}L)$. In general, for any ideal I of R and any R-modules M_1, M_2, the following hold:

- $(M_1\cap M_2):I=(M_1:I)\cap(M_2:I)$,
- $I(M_1\cap M_2)\subseteq IM_1\cap IM_2$,
- If $M_1\subseteq M_2$, then $M_1:I\subseteq M_2:I$.

Hence we have:

$$N\cap L\subseteq(\mathfrak{m}(N\cap L):l)\subseteq(\mathfrak{m}N\cap\mathfrak{m}L):l=(\mathfrak{m}N:l)\cap(\mathfrak{m}L:l)=N\cap L.$$

Therefore $\mathfrak{m}(N\cap L):l=N\cap L$. This implies $N\cap L\in\mathfrak{m}\text{-Full}(M)$.

(5): From (1) and (3), $D(N)\cap D(L)\neq\emptyset$. So take an element $l\in D(\mathfrak{m}N)\cap D(\mathfrak{m}L)$. We have:

$$(N\cap L):l=(N:l)\cap(L:l)=(N:\mathfrak{m})\cap(L:\mathfrak{m})=(N\cap L):\mathfrak{m}.$$

This implies $N\cap L\in\mathrm{Full}(M)$. □

Definition 5. *Let* $N\in\mathrm{sub}(M)$.

(1) *If there exists an unique minimal element (w.r.t. inclusion order) among those elements* $L\supseteq N$ *with* $L\in\mathfrak{m}\text{-Full}(M)$, *then we call it the "* $\mathfrak{m}$ *-full closure of* N *" and denote it by* $\widetilde{N}^{\mathfrak{m}\text{-}f}$.

(2) *Similarly, if there exists an unique minimal element (w.r.t. inclusion order) among those elements* $L\supseteq N$ *with* $L\in\mathrm{Full}(M)$, *then we call it the "full closure of* N *" and denote it by* $\widetilde{N}^{f}$.

Theorem 3. *For any* $N\in\mathrm{Full}(M)$, *the following hold:*

(1) *There exist* $\widetilde{N}^{\mathfrak{m}\text{-}f}$ *the* $\mathfrak{m}$ *-full closure of* N.

(2) *There exist* $\widetilde{N}^{f}$ *the full closure of* N.

Proof. (1) :Since $l(M/N)<\infty$ for $N\in\mathrm{sub}(M)$, $\Omega(N):=\{L\in\mathrm{sub}(M) | L\supseteq N\}$ satisfies the decending chain condition (w.r.t. inclusion order) . We have $M\in\Omega(N)\cap\mathfrak{m}\text{-Full}(M)\neq\emptyset$, so there exists a minimal element (w.r.t. inclusion order) in $\Omega(N)\cap\mathfrak{m}\text{-Full}(M)$. If L,L' are minimal elements (w.r.t. inclusion order) in $\Omega(N)\cap\mathfrak{m}\text{-Full}(M)$, then $L\cap L'\in\mathfrak{m}\text{-Full}(M)$. Hence $L=L\cap L'=L'$ form the minimality of these elements. This implies $\left|\min\{\Omega(N)\cap\mathfrak{m}\text{-Full}(M)\}\right|=1$, where $|S|$ denotes the cardinality of a set S. Therefore $\widetilde{N}^{\mathfrak{m}\text{-}f}$ the $\mathfrak{m}$-full closure of N exists.

(2):The proof of (2) is quite similar to the proof of (1). So we omit the proof. □

4. Restricted weak Lefschetz property

4.1 Preliminary

Let W be a finte dimensional k-vector space with surejective linear maps: $W = W_n \underset{f_n}{\twoheadrightarrow} W_{n-1} \twoheadrightarrow \cdots \underset{f_2}{\twoheadrightarrow} W_1 \underset{f_1}{\twoheadrightarrow} 0$.

Lemma 7. $\dim_k W = \dim_k \bigoplus_{i=1}^{n} \ker\left(W_i \underset{f_i}{\twoheadrightarrow} W_{i-1} \right)$.

Proof. We prove this by induction on n. If $n=1$, the assertion clearly holds. If $n>1$, then we have $\dim_k W_{n-1} = \dim_k \bigoplus_{i=1}^{n-1} \ker\left(W_{i-1} \underset{f_i}{\twoheadrightarrow} W_i \right)$ by the induction hypothesis. Therefore we have:

$$\dim_k W_n = \dim_k \ker\left(W_n \underset{f_n}{\twoheadrightarrow} W_{n-1} \right) + \dim_k W_{n-1} = \dim_k \ker\left(W_n \underset{f_n}{\twoheadrightarrow} W_{n-1} \right) + \dim_k \bigoplus_{i=1}^{n-1} \ker\left(W_{i-1} \underset{f_i}{\twoheadrightarrow} W_i \right) = \dim_k \bigoplus_{i=1}^{n} \ker\left(W_{i-1} \underset{f_i}{\twoheadrightarrow} W_i \right). \quad \square$$

4.2 Restricted weak Lefschetz property implies weak fullness

Lemma 8. *For any* $N \in \mathrm{sub}(M)$*, the following hold:*

(1) *If* N *has restricted weak Lefschetz property w.r.t.* M*, then* $N \in \text{w-Full}(M)$.

(2) *If* $\mathfrak{m}N$ *has restricted weak Lefschetz property w.r.t.* M*, then* $N \in \text{w-}\mathfrak{m}\text{-Full}(M)$.

Proof. (1) :Denote $M/N = \bigoplus_{i=a}^{s} (M/N)_i$ *with* $(M/N)_a, (M/N)_s \neq 0$, then by our assumption, M/N has unimodal Hilbert function. So we have $0 = h_{M/N}(a-1) \leq h_{M/N}(a) \leq \cdots \leq h_{M/N}(\sigma) \geq h_{M/N}(\sigma+1) \geq \cdots \geq h_{M/N}(s) \geq h_{M/N}(s+1) = 0$ for some integer c with $a \leq c \leq s$. Hence we have:

$$(\#) \quad 0 = (M/N)_{a-1} \underset{\times l}{\hookrightarrow} (M/N)_a \underset{\times l}{\hookrightarrow} \cdots \underset{\times l}{\hookrightarrow} (M/N)_c \underset{\times l}{\twoheadrightarrow} \cdots \underset{\times l}{\twoheadrightarrow} (M/N)_s \underset{\times l}{\twoheadrightarrow} (M/N)_{s+1} = 0,$$

where $l \in R_1$ is a Lefschetz element, $\times l$ denote the linear map defined by multiplication by l, $\hookrightarrow$'s are injective maps and $\twoheadrightarrow$'s are surjective maps.

From (#) and Lemma 7, we have:

$$(\natural) \quad \dim_k (M/N)_c = \dim_k \bigoplus_{i \geq c} \ker\left((M/N)_i \underset{\times l}{\twoheadrightarrow} (M/N)_{i+1} \right) = l\left(\ker\left((M/N)_{\geq c} \underset{\times l}{\twoheadrightarrow} (M/N)_{\geq c} \right) \right) = l\left(\ker\left(M/N \underset{\times l}{\rightarrow} M/N \right) \right) = t_l(N),$$

where we denote $(M/N)_{\geq c} := \bigoplus_{i \geq c} (M/N)_i$.

Put $N' = \bigoplus_{i \leq c} N_i \oplus \bigoplus_{i \geq c+1} M_i \supseteq N$, where N_i (resp. M_i) denotes the the graded componet of N (resp. M) for any integer *i*, then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \rightarrow & (N'/N)_i & \rightarrow & (M/N)_i & \rightarrow & (M/N')_i & \rightarrow & 0\ (\text{exact}), \\
 & & \downarrow \times l & \circlearrowleft & \downarrow \times l & \circlearrowleft & \downarrow \times l & & \\
0 & \rightarrow & (N'/N)_i & \rightarrow & (M/N)_i & \rightarrow & (M/N')_i & \rightarrow & 0\ (\text{exact}).
\end{array}$$

Since $(N'/N)_i = 0$, for any integer $i \leq c$ and $(N'/N)_i = (M/N)_i$ for any integer $i > c$ in the above diagram, we have:

$$(\natural\natural) \quad (M/N)_c \simeq (M/N')_c = (M/N')_{\geq c} \subseteq (N' : \mathfrak{m})/N'.$$

From ($\natural$) and ($\natural\natural$), we get

$$\tau(N') = l((N' : \mathfrak{m})/N') \geq l(M/N')_{\geq c} = \dim_k (M/N)_c = t_l(N).$$

Therefore, by Lemma 2(8), we have $\tau(N') = t_l(N) (= t(N))$. Hence, by Lemma 4, $N \in \text{w-Full}(M)$.

(2) :This follows from (1) and Lemma 3. $\square$

Theorem 4. *For any* $N \in \mathrm{sub}(M)$*, the following hold:*

(1) *If* N *has restricted weak Lefschetz property w.r.t.* M*, then the following two conditions are equivalent:*

a) $N \in \mathrm{SRees}(M)$.

b) $N \in \mathrm{Full}(M)$.

(2) *If* $\mathfrak{m}N$ *has restricted weak Lefschetz property w.r.t.* M*, then the following two conditions are equivalent:*

a) $N \in \mathrm{Rees}(M)$.

b) $N \in \mathfrak{m}\text{-Full}(M)$.

Proof. (1) : This follows from Theorem 2(3) and Lemma 8(1).

(2) : This follows from Theorem 2(2) and Lemma 8(2). $\square$

5. Behavior of Rees property and its related properties under the inclusion-reversing bijection

5.1 Preliminary

Lemma 9. *For any ideal* I *of* R *and any R-module* M *, the following are equivalent:*

a) $\alpha \in M$ *and* $\varphi(\alpha)=0$ *for all* $\varphi \in I(M^{\vee})$.

b) $\alpha \in 0 \underset{M}{:} I$.

Proof. a) $\Rightarrow$ b): For any $r \in I$, we have $r\varphi(\alpha)=\varphi(r\alpha)=0$ $\left({}^{\forall}\varphi \in M^{\vee}\right)$ by the assumption and the definition of R-module structure of $M^{\vee}$. This implis $r\alpha = 0$ for any $r \in I$. Hence we have $\alpha \in L:I$.

b) $\Rightarrow$ a): Any $\varphi \in I(M^{\vee})$ can be represented in the form: $\varphi = \sum_{j=1}^{m} r_j \varphi_j$ for some $r_j \in I$ and $\varphi_j \in L^{\vee}$ ($0 \le j \le m$).

Therefore $\varphi(\alpha)=\sum_{j=1}^{m} r_j \varphi_j(\alpha)=\sum_{j=1}^{m}\varphi_j(r_j\alpha)=0$ for all $\varphi \in I(L^{\vee})$. □

5.2 Behavior of Rees property and its related properties under the inclusion-reversing bijection $(\ )^{\circ}$

In this section, we assume that M is a graded R-module of finite length, i.e., $l(M)<\infty$.

Lemma 10. *For any* $N \in \mathrm{sub}(M)$*, the following hold:*

(1) $(N^{\circ})^{\circ}=N$ *under the natural identification* $M \underset{\text{id.}}{=} (M^{\vee})^{\vee}\left(\supseteq (N^{\circ})^{\circ}\right)$.

(2) $I(N^{\circ})=(N:I)^{\circ}$.

(3) $N^{\circ}:I=(IN)^{\circ}$.

(4) $L^{\circ}/N^{\circ} \simeq (L/N)^{\vee}$ *if* $N \subseteq L \in \mathrm{sub}(M)$.

Proof. (1):It is easy to check, so we omit the proof.

(2): We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & \ker(\pi\circ\varepsilon) & \to & M/N & & & & (\text{exact}), \\
 & & \downarrow \iota & \circlearrowleft & \downarrow \varepsilon & & & & \\
0 & \to & \left(\dfrac{(M/N)^{\vee}}{I(M/N)^{\vee}}\right)^{\vee} & \to & \left((M/N)^{\vee}\right)^{\vee} & \overset{\pi}{\to} & \left(I(M/N)^{\vee}\right)^{\vee} & \to & 0\ (\text{exact}),
\end{array}$$

where $\varepsilon : M/N \to \left((M/N)^{\vee}\right)^{\vee}$ is the evaluation map, defined by $\varepsilon(\alpha)(\varphi):=\varphi(\alpha)$ for $\alpha \in M/N$ and $\varphi \in (M/N)^{\vee}$.

Since the map ε is an isomorphism, we have $\left(\dfrac{(M/N)^{\vee}}{I(M/N)^{\vee}}\right)^{\vee} \simeq \ker(\pi\circ\iota)$. Furthermore, by Lemma 9, we have:

$$\alpha \in \ker(\pi\circ\iota)(\subseteq M/N) \Leftrightarrow \varepsilon(\alpha)\big|_{I(M/N)^{\vee}}=0 \Leftrightarrow \varphi(\alpha)=0 \text{ for } {}^{\forall}\varphi \in I(M/N)^{\vee} \Leftrightarrow \alpha \in 0 \underset{M/N}{:} M/N = N:I/N.$$

This implies $\left(\dfrac{(M/N)^{\vee}}{I(M/N)^{\vee}}\right)^{\vee} \simeq \ker(\pi\circ\iota)=N:I/N$. Taking $(\ )^{\vee}$ of the above diagram , we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & I(M/N)^{\vee} & \to & (M/N)^{\vee} & \to & \dfrac{(M/N)^{\vee}}{I(M/N)^{\vee}} & \to & 0\,(\text{exact}), \\
 & & \downarrow & \circlearrowleft & \|\varepsilon^{\vee} & \circlearrowleft & \downarrow \iota^{\vee} & & \\
0 & \to & (M/N:I)^{\vee} & \to & (M/N)^{\vee} & \to & (N:I/N)^{\vee} & \to & 0\ (\text{exact}).
\end{array}$$

Therefore we get $I(N^{\circ})=I(M/N)^{\vee}=(M/N:I)^{\vee}=(N:I)^{\circ}$ in $(M/N)^{\vee}$ from the five lemma.

(3):In (2), replacing M by $M^{\vee}$ and N by N°, we have $IN = I\left((N^{\circ})^{\circ}\right)=(N^{\circ}:I)^{\circ}$. Hence we have $(IN)^{\circ}=\left((N^{\circ}:I)^{\circ}\right)^{\circ}=N^{\circ}:I$.

(4): Since we have $L^{\circ}=(M/L)^{\vee}$ and $N^{\circ}=(M/N)^{\vee}$, (4) follows from the exact sequence:

$$0 \to (M/L)^{\vee} \to (M/N)^{\vee} \to (L/N)^{\vee} \to 0 \quad (\text{exact}). \qquad \square$$

From Lemma 10, $(\ )^{\circ}:\mathcal{P}=\mathrm{sub}(M)\to\mathcal{P}'=\mathrm{sub}(M^{\vee})$ (resp. $(\ )^{\dagger}=(\ )^{\circ}:\mathcal{P}=\mathrm{sub}(M)\to\mathcal{Q}=\mathrm{sub}(M^{\vee})$) satisfies Condition 5(resp. Conditon2) in [6]. Therefore we can apply the results in 2.4 and 3.5 in [6] to $(\ )^{\circ}:\mathcal{P}=\mathrm{sub}(M)\to\mathcal{P}'=\mathrm{sub}(M^{\vee})$.

We denote

$$u_l(N):=l(N/lN) \quad\text{and}\quad u_M(N)=u(N):=\min_{l\in R_1}u_l(N).$$

Let $q^{\#}N^{\circ}, q_{\#}N^{\circ}, q^{l}N^{\circ}, q_{l}N^{\circ}, \min q_{\Phi}N^{\circ}$ and $\min q^{\Phi}N^{\circ}$ be the notations in 3.5 in [6] where $N^{\circ}\in\mathcal{P}'=\mathrm{sub}(M^{\vee})$ and $\Phi=R_1$ (resp. $q^{\flat}N^{\circ}$ and $q_{\flat}N^{\circ}$ be the notations in 2.4 in [6] where $N^{\circ}\in\mathcal{Q}=\mathrm{sub}(M^{\vee})$). It is easy to check that the following hold by using Lemma 5, Lemma 2 in [6] and Lemma 10:

- $q^{\#}N^{\circ}\left(=q^{\flat}N^{\circ}\right)=\tau(N)=\mu(N^{\circ})$, $q_{\#}N^{\circ}\left(=q_{\flat}N^{\circ}\right)=\mu(N)=\tau(N^{\circ})$.
- $q^{l}N^{\circ}=t_l(N)=u_l(N^{\circ})$, $q_{l}N^{\circ}=u_l(N)=t_l(N^{\circ})$.
- $\min q_{\Phi}N^{\circ}=u(N)=t(N^{\circ})$, $\min q^{\Phi}N^{\circ}=t(N)=u(N^{\circ})$.

Theorem 5. *For any* $N\in\mathrm{sub}(M)$, *the following hold:*

(1) $N^{\circ}\in\mathrm{Rees}(M)$ *if and only if* $\overline{\tau}(N)=\tau(N)$.

(2) $N^{\circ}\in\mathrm{SRees}(M)$ *if and only if* $\overline{\mu}(N)=\mu(N)$.

(3) $N^{\circ}\in\mathfrak{m}\text{-Full}(M)$ *if and only if* $l(N:\mathfrak{m})=N$ *for some* $l\in R_1$.

(4) $N^{\circ}\in\text{w-}\mathfrak{m}\text{-Full}(M)$ *if and only if* $\overline{\tau}(N)\geq u(N:\mathfrak{m})$.

(5) $N^{\circ}\in\mathrm{Full}(M)$ *if and only if* $\mathfrak{m}N=lN$ *for some* $l\in R_1$.

(6) $N^{\circ}\in\text{w-Full}(M)$ *if and only if* $\overline{\mu}(N)=u(N)$.

Proof. (1), (2): These follow from Proposition 2 in [6].

(3)-(6): These follow from Proposition 4 in [6].

Remark 2. *If we replace the ground ring R a Noetherian standard graded commutative algebra over a field* k *by a Noetherian local ring with maximal ideal* $\mathfrak{m}$ *and replace* $\Phi=R_1$ by $\mathfrak{m}\setminus\mathfrak{m}^2$, *the results in this paper are still valid except the results in section 4.*

Acknowledgement. The author would like to thank Professor Tadahito Harima and Professor Junzo Watanabe for their helpful comments and discussions.

(Received: Sep. 25, 2014)
(Accepted: Dec. 3, 2014)

References

[1] A. Conca, E. De Negri and M. E. Rossi: *Integrally closed and componentwise linear ideals*, Math. Z. **265**, pp.197-210, (2010).
[2] J. Hong, H. Lee, S. Noh and D. E. Rush: *Full ideals*, Comm. Algebra 37, pp. 2627-2639, (2009).
[3] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi, J. Watanabe: *The Lefschetz Properties*, Lecture Notes in Mathematics **2080**, Springer.
[4] T. Harima and J. Watanabe: *The weak Lefschetz property for m-full ideals and componentwise linear ideals*, Illinois J. Math. **56**, pp. 957-966, (2012).
[5] T. Harima and J. Watanabe: *Completely m-full ideals and componentwise linear ideals*, preprint.
[6] S. Isogawa: *Rees property and its related properties of ranked partially ordered sets*, preprint (to appear in this volume).
[7] J. Migliore, R. Mir-Roig, S. Murai, U. Nagel and J. Watanabe: *On ideals with the Rees property*, Archiv Math. **101**, pp. 445-454, (2013).
[8] J. Watanabe: *The Dilworth number of Artinian rings and finite posets with rank function*, Commutative Algebra and Combinatorics, Advanced Studies in Pure Math. Vol. **11**, Kinokuniya Co. North Holland, Amsterdam, pp.303-312, (1987).
[9] J. Watanabe: *m-Full Ideals*, Nagoya Math. Journal, **106**, pp. 101-111, (1987).
[10] J. Watanabe: *The syzygies of m-full ideals*, Math. Proc. Cambridge Philos. Soc. **109**, pp. 7-13, (1991).
[11] J. Watanabe: *m-Full Ideals II*, Math. Proc. Cambridge Philos. Soc. **111**, pp. 231-240, (1992).