Paper

Rees property and its related properties of ranked partially ordered sets

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Abstract Properties of ideals of a commutative Noetherian local ring or a Noetherian standerd graded commutative algebra over a field, called the Rees property, the second Rees property, the m-fullness and the fullness, can naturally be extended to the properties of ranked partially ordered sets. In this paper we study these properties with purely combinatorial points of view.

Keywords : Rees property, Second Rees property, fullness, m-fullness, Ranked partially ordered set.

1. Introduction

Properties of ideals of a commutative Noetherian local ring or a Noetherian standerd graded commutative algebra over a field, called the Rees property, the second Rees property, the fullness and the m-fullness, have been studied by many authors (e.g., [1]-[9]). Especially the notion of the second Rees property has been introduced by J. Watanabe in a private conversation with T. Harima in 2008.

In this paper, we naturally extend these properties of ideals to the properties of ranked partially ordered sets. More precisely, these properties can be denoted by using order preserving maps and the rank function on partially ordered set. We study these properties with purely combinatorial points of view.

In section 2, we establish a correspondence between elements having the Rees property and those having restricted second Rees property. Similarly in section 3, a correspondence between elements having the m-fullness and those having the restricted fullness is given. In section 4, we study the inclusion relations which hold among subsets having those properties the Rees property, the second Rees property, the fullness and the m-fullness. Moreover, we introduce weak m-fullness to give a condition where the Rees property and the m-fullness coincide, and also introduce weak fullness to give a condition where the second Rees property and the fullness coincide.

2. A correspondence between the Rees subset and the restricted second Rees subset

2.1 Main setting

Let $\mathcal{P}=(\mathcal{P},\leq)$ be a partially ordered set, $\mathbb{Z}$ be the set of integers and $\mathbb{Z}_{\geq 0}$ be the set of nonnegative integers. We denote $\mathrm{Map}(\mathcal{P},\mathbb{Z})$ the set of maps from $\mathcal{P}$ to $\mathbb{Z}$. First we recall the definition of a rank function on $\mathcal{P}$. An order preserving map $r\in\mathrm{Map}(\mathcal{P},\mathbb{Z})$, is called a *rank function*, if it satisfies the condition that $x=y$ whenever $x\leq y$ and $r(x)=r(y)$. In general, a partially ordered set with a rank function is called a *ranked poset*. From now on, we assume that $\mathcal{P}$ is a ranked poset with a rank function r and assume that $(\ )^{\#},(\ )_{\#}:\mathcal{P}\to\mathcal{P}$ are two order preserving maps satisfying the following conditions:

Condition 1. *For any* $\alpha\in\mathcal{P}$,

$$\alpha_{\#}\leq(\alpha^{\#})_{\#}\leq\alpha\leq(\alpha_{\#})^{\#}\leq\alpha^{\#}.$$

Remark 1. *The above conditions make* $((\ )^{\#},(\ )_{\#})$ *an adjoint pair when we think of* $\mathcal{P}$ *and* $(\ )^{\#},(\ )_{\#}:\mathcal{P}\to\mathcal{P}$ *as a category and functors. Moreover in this case, we have*

$$((\alpha^{\#})_{\#})^{\#}=\alpha^{\#} \quad\text{and}\quad ((\alpha_{\#})^{\#})_{\#}=\alpha_{\#}.$$

2.2 Preliminary lemma

Definition 1. *We define two functions* $r^{\#}, r_{\#}\in\mathrm{Map}(\mathcal{P},\mathbb{Z})$ *as follows:*

$$r^{\#}\alpha:=r\alpha^{\#}-r\alpha \quad and \quad r_{\#}\alpha:=r\alpha-r\alpha_{\#}$$

where r *is the rank function on* $\mathcal{P}$.

For any $f\in\mathrm{Map}(\mathcal{P},\mathbb{Z})$, we denote

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$$\overline{f}(\alpha) := \max\{ f(x) | x \ge \alpha \} \quad \text{(if it exists, otherwise } \overline{f}(\alpha) := \infty \text{)},$$
$$\underline{f}(\alpha) := \max\{ f(x) | x \le \alpha \} \quad \text{(if it exists, otherwise } \underline{f}(\alpha) := \infty \text{)}.$$

We put $\mathcal{P}^\# := \{\alpha^\# | \alpha \in \mathcal{P}\}$ and $\mathcal{P}_\# := \{\alpha_\# | \alpha \in \mathcal{P}\}$.

Lemma 1. *Let α, β be elements of $\mathcal{P}$. The following hold:*

(1) $(\alpha_\#)^\# = \alpha$ *if and only if* $\alpha \in \mathcal{P}^\#$, $(\alpha^\#)_\# = \alpha$ *if and only if* $\alpha \in \mathcal{P}_\#$.

(2) *If* $\alpha \le \beta$, *then* $r_\#\alpha \le \underline{r}_\#\beta$, $r^\#\alpha \le \underline{r}^\#\beta$.

(3) $r^\#\alpha \le r_\#\alpha^\# = r^\#(\alpha^\#)_\#$. *Especially* $r^\#\alpha = r_\#\alpha^\#$ *if and only if* $(\alpha^\#)_\# = \alpha$.

(4) $r_\#\alpha \le r^\#\alpha_\# = r_\#(\alpha_\#)^\#$. *Especially* $r_\#\alpha = r^\#\alpha_\#$ *if and only if* $(\alpha_\#)^\# = \alpha$.

(5) $\underline{r}^\#\alpha = \underline{r}_\#\alpha^\#$.

(6) $\underline{r}_\#\alpha = \underline{r}^\#\alpha_\#$ *if* $(\alpha_\#)^\# = \alpha$.

(7) *If* $r^\#\alpha = \underline{r}^\#\alpha$, *then* $r_\#\alpha^\# = \underline{r}_\#\alpha^\#$ *and* $(\alpha^\#)_\# = \alpha$.

(8) *If* $r_\#\alpha = \underline{r}_\#\alpha$ *and* $\alpha \in \mathcal{P}^\#$, *then* $r^\#\alpha_\# = \underline{r}^\#\alpha_\#$ *and* $(\alpha_\#)^\# = \alpha$.

(9) $r_\#\alpha = \underline{r}_\#\alpha$ *and* $\alpha \in \mathcal{P}^\#$ *if and only if* $r_\#\alpha = \underline{r}_\#(\alpha_\#)^\#$.

Proof. (1) :By Remark 1, it is easy to check. (2) :This directly follows from the definitions.

(3): Since $(\alpha^\#)_\# \le \alpha$, we have

$$r_\#\alpha^\# - r^\#\alpha = \left(r\alpha^\# - r(\alpha^\#)_\#\right) - \left(r\alpha^\# - r\alpha\right) = r\alpha - r(\alpha^\#)_\# \ge 0.$$

Equality holds if and only if $(\alpha^\#)_\# = \alpha$.

(4) :We can prove simiarly as (3).

(5): If $\underline{r}^\#\alpha = \infty$, then for any $n \in \mathbb{Z}_{\ge 0}$ there exists $\alpha_n \le \alpha$ such that $n \le r^\#\alpha_n$. Since $\alpha_n{}^\# \le \alpha^\#$ and $n \le r^\#\alpha_n \le r_\#\alpha_n{}^\#$ by (3), we have $\underline{r}_\#\alpha^\# = \infty$. On the contrary, if $\underline{r}_\#\alpha^\# = \infty$, then for any $n \in \mathbb{Z}_{\ge 0}$ there exists $\beta_n \le \alpha^\#$ such that $n \le r_\#\beta_n$. Since $(\beta_n)_\# \le (\alpha^\#)_\# \le \alpha$ and $n \le r_\#\beta_n \le r^\#(\beta_n)_\#$ by (4), we have $\underline{r}^\#\alpha = \infty$. Therefore we assume $\underline{r}^\#\alpha, \underline{r}_\#\alpha^\# < \infty$. If $\underline{r}^\#\alpha = r^\#\beta$ for some $\beta \le \alpha$, then $\beta^\# \le \alpha^\#$ and $\underline{r}^\#\alpha = r^\#\beta \le r_\#\beta^\# \le \underline{r}_\#\alpha^\#$ by (3) and (2). Similarly, if $\underline{r}_\#\alpha^\# = r_\#\gamma$ for some $\gamma \le \alpha^\#$, then $\gamma_\# \le (\alpha^\#)_\# \le \alpha$ and $\underline{r}_\#\alpha^\# = r_\#\gamma \le r^\#\gamma_\# \le \underline{r}^\#\alpha$ by (4) and (2).

(6) :From (5), replacing α by $\alpha_\#$, we have $\underline{r}^\#\alpha_\# = \underline{r}_\#(\alpha_\#)^\# = \underline{r}_\#\alpha$.

(7): We have $r^\#\alpha \le r_\#\alpha^\# = r^\#(\alpha^\#)_\# \le \underline{r}^\#\alpha = r^\#\alpha$, therefore $r^\#\alpha = r_\#\alpha^\#$ and $(\alpha^\#)_\# = \alpha$ by (3). Moreover from (6) and (3), we get $\underline{r}_\#\alpha^\# = \underline{r}^\#(\alpha^\#)_\# = \underline{r}^\#\alpha = r^\#\alpha = r_\#\alpha^\#$.

(8): We remark $(\alpha_\#)^\# = \alpha$ by (1) and $r_\#\alpha = r^\#\alpha_\#$ by (4). From (5) and (4), we have $\underline{r}^\#\alpha_\# = \underline{r}_\#(\alpha_\#)^\# = \underline{r}_\#\alpha = r_\#\alpha = r^\#\alpha_\#$.

(9): First we assume $r_\#\alpha = \underline{r}_\#(\alpha_\#)^\#$, then $r^\#\alpha_\# \le r_\#(\alpha_\#)^\# \le \underline{r}_\#(\alpha_\#)^\# = r_\#\alpha \le r^\#\alpha_\#$. This implies $r_\#\alpha = r^\#\alpha_\#$. From (4), we have $(\alpha_\#)^\# = \alpha$. Therefore $r_\#\alpha = \underline{r}_\#(\alpha_\#)^\# = \underline{r}_\#\alpha$ and $\alpha \in \mathcal{P}^\#$ by (1). Hence "if" part follows. Since $\alpha \in \mathcal{P}^\#$ implies $(\alpha_\#)^\# = \alpha$ by (1), the converse implication follows. □

We denote

$$d(\mathcal{P}) := \max\{ r^\#\alpha | \alpha \in \mathcal{P}\} \quad \text{(if it exists, otherwise } d(\mathcal{P}) := \infty \text{)},$$
$$d'(\mathcal{P}) := \max\{ r_\#\alpha | \alpha \in \mathcal{P}\} \quad \text{(if it exists, otherwise } d'(\mathcal{P}) := \infty \text{)}$$

Proposition 1. $d(\mathcal{P}) = d'(\mathcal{P})$.

Proof. If $d(\mathcal{P}) = \infty$, then there exists $\alpha_n \in \mathcal{P}$ wiht $n \le r^\#\alpha_n$ for each $n \in \mathbb{Z}_{\ge 0}$. By Lemma 1(3), $r^\#\alpha_n \le r_\#\alpha_n$. Hence we have $d'(\mathcal{P}) = \infty$. Similarly, $d(\mathcal{P}) = \infty$ if $d'(\mathcal{P}) = \infty$. Therefore we assume that $d(\mathcal{P}), d'(\mathcal{P}) < \infty$. If $d(\mathcal{P}) = r^\#\alpha$ for some $\alpha \in \mathcal{P}$, then $d(\mathcal{P}) = r^\#\alpha \le r_\#\alpha^\# \le d'(\mathcal{P})$ again by Lemma 1(3). Conversely, if $d'(\mathcal{P}) = r_\#\beta$ for some $\beta \in \mathcal{P}$, then $d'(\mathcal{P}) = r_\#\beta \le r^\#\beta_\# \le d(\mathcal{P})$ by Lemma 1(4). Hence $d(\mathcal{P}) = d'(\mathcal{P})$. □

2.3 The Rees subset, the restricted second Rees subset and their correspondence

For any maps $f, g \in \mathrm{Map}(\mathcal{P}, \mathbb{Z} \cup \{\infty\})$, we denote the equalizer of these maps by

$$\mathrm{Eql}_{\mathcal{P}}(f, g) = \mathrm{Eql}(f, g) := \{\alpha \in \mathcal{P} \mid f(\alpha) = g(\alpha)\}.$$

Definition 2. *We define three subsets* $\mathrm{Rees}\mathcal{P}$, $\mathrm{SRees}\mathcal{P}$ *and* $\mathrm{SRees}'\mathcal{P}$ *of* $\mathcal{P}$ *as follows:*

$$\mathrm{Rees}\mathcal{P} := \mathrm{Eql}(r^{\#}, \underline{r^{\#}}),\ \mathrm{SRees}\mathcal{P} := \mathrm{Eql}(r_{\#}, \underline{r_{\#}}) \ \ \textit{and} \ \ \mathrm{SRees}'\mathcal{P} := \mathrm{SRees}\mathcal{P} \cap \mathcal{P}^{\#},$$

and we call them "Rees subset", "second Rees subset" and "restricted second Rees subset" of $\mathcal{P}$ *respectively.*

Remark 2. *From Lemma 1(9), we have* $\mathrm{SRees}'\mathcal{P} = \mathrm{Eql}\left(r_{\#}, \underline{r_{\#} \circ ((\ )_{\#})^{\#}}\right)$.

We can establish a one-to-one correspondence between $\mathrm{Rees}\mathcal{P}$ and $\mathrm{SRees}'\mathcal{P}$. We state this bellow as a theorem.

Theorem 1. *There is an one- to-one correspondence:*

$$\mathrm{Rees}\mathcal{P} \underset{(\ )_{\#}}{\overset{(\ )^{\#}}{\rightleftarrows}} \mathrm{SRees}'\mathcal{P}.$$

Proof. This follows immediately from Lemma 1(7),(8). □

2.4 Behavior of these properties under an order reversing bijection

At the end of this section, we describe how these properties behave under an order reversing bijection with some conditions. Let $\mathcal{Q}$ be a poset with order preserving maps $(\ )^{\flat}, (\ )_{\flat} : \mathcal{Q} \to \mathcal{Q}$, which satisfy $\beta_{\flat} \le (\beta^{\flat})_{\flat} \le \beta \le (\beta_{\flat})^{\flat} \le \beta^{\flat}$ for any $\beta \in \mathcal{Q}$. Let $(\ )^{\dagger} : \mathcal{P} \to \mathcal{Q}$ be an order reversing bijection which satisfies the following conditions:

Condition 2. *For any* $\alpha \in \mathcal{P}$,

$$(\alpha^{\dagger})^{\flat} = (\alpha_{\#})^{\dagger} \ \ \textit{and} \ \ (\alpha^{\dagger})_{\flat} = (\alpha^{\#})^{\dagger}.$$

If we put $q\alpha^{\dagger} := -r\alpha$, then $\mathcal{Q}$ becomes a ranked poset with the rank function q. We denote $q^{\flat}\beta := q\beta^{\flat} - q\beta$ and $q_{\flat}\beta := q\beta - q_{\flat}\beta$ for all $\beta \in \mathcal{Q}$.

Under these conditions, we have the following lemma and proposition.

Lemma 2. *The following hold for any* $\alpha \in \mathcal{P}$:

(1) $q^{\flat}\alpha^{\dagger} = r_{\#}\alpha$, $q_{\flat}\alpha^{\dagger} = r^{\#}\alpha$

(2) $\underline{q^{\flat}}\alpha^{\dagger} = \overline{r_{\#}}\alpha$, $\underline{q_{\flat}}\alpha^{\dagger} = \overline{r^{\#}}\alpha$.

Proof. (1): From the definitions we have

$$q^{\flat}\alpha = q((\alpha^{\dagger})^{\flat}) - q\alpha^{\dagger} = q(\alpha_{\#})^{\dagger} + r\alpha = r\alpha - r\alpha_{\#} = r_{\#}\alpha, \ \ q_{\flat}\alpha^{\dagger} = q\alpha^{\dagger} - q((\alpha^{\dagger})_{\flat}) = -r\alpha - q((\alpha^{\#})^{\dagger}) = r\alpha^{\#} - r\alpha = r^{\#}\alpha.$$

(2): From the definitions we have

$$\underline{q^{\flat}}\alpha^{\dagger} = \max\left\{q^{\flat}\beta = r_{\#}\alpha' \,\middle|\, (\alpha')^{\dagger} = \beta \le \alpha^{\dagger}\right\} = \max\{r_{\#}\alpha' \mid \alpha' \ge \alpha\} = \overline{r_{\#}}\alpha.$$

$$\underline{q_{\flat}}\alpha^{\dagger} = \max\left\{q_{\flat}\beta = r^{\#}\alpha' \,\middle|\, (\alpha')^{\dagger} = \beta \le \alpha^{\dagger}\right\} = \max\{r^{\#}\alpha' \mid \alpha' \ge \alpha\} = \overline{r^{\#}}\alpha. \ \ \square$$

Proposition 2. *For any* $\alpha \in \mathcal{P}$, *the following hold:*

(1) $\alpha^{\dagger} \in \mathrm{Rees}\mathcal{Q}$ *if and only if* $\overline{r_{\#}}\alpha = r_{\#}\alpha$.

(2) $\alpha^{\dagger} \in \mathrm{SRees}\mathcal{Q}$ *if and only if* $\overline{r^{\#}}\alpha = r^{\#}\alpha$.

Proof. (1): Since $q^{\flat}\alpha^{\dagger} = r_{\#}\alpha$ and $\underline{q^{\flat}}\alpha^{\dagger} = \overline{r_{\#}}\alpha$, $\alpha^{\dagger} \in \mathrm{Rees}\mathcal{Q} = \mathrm{Eql}_{\mathcal{Q}}(q^{\flat}, \underline{q^{\flat}})$ if and only if $\overline{r_{\#}}\alpha = r_{\#}\alpha$.

(2): Similarly, since $q_{\flat}\alpha^{\dagger} = r^{\#}\alpha$ and $\underline{q_{\flat}}\alpha^{\dagger} = \overline{r^{\#}}\alpha$, $\alpha^{\dagger} \in \mathrm{SRees}\mathcal{Q} = \mathrm{Eql}_{\mathcal{Q}}(q_{\flat}, \underline{q_{\flat}})$ if and only if $\overline{r^{\#}}\alpha = r^{\#}\alpha$. □

3. A correspondence between the m-full subset and the restrictedly full subset

3.1 Main setting

In this section, we need more additional assumptions on $\mathcal{P}$. We assume that there is an infimum $\alpha \wedge \beta \in \mathcal{P}$ of each pair of two elements $\alpha, \beta \in \mathcal{P}$. This means that $\alpha \wedge \beta \le \alpha, \beta$ and $\gamma \le \alpha \wedge \beta$ whenever $\gamma \le \alpha, \beta$. We introduce a family of order preserving maps $(\ )^{l}, (\ )_{l} : \mathcal{P} \to \mathcal{P}\ (l \in \Phi)$ indexed by a set Φ, which satisfies the following conditions:

Condition 3.

(1) *For any* $\alpha \in \mathcal{P}$ *and* $l \in \Phi$, $\alpha_{l} \le (\alpha^{l})_{l} \le \alpha \le (\alpha_{l})^{l} \le \alpha^{l}$ *and* $\alpha_{l} \le \alpha_{\#} \le \alpha^{\#} \le \alpha^{l}$.

(2) *For any* $\alpha \in \mathcal{P}$ *and* $l \in \Phi$, $r_{l}\alpha = r(\alpha \wedge \beta^{l}) - r\beta$ *whenever* $\beta \le \alpha_{l}$, *where we denote* $r_{l}\alpha := r\alpha - r_{l}\alpha$.

(3) *For any* $\alpha \in \mathcal{P}$ *and* $l_1, l_2 \in \Phi \cup \{\#\}$, $(\alpha_{l_1})_{l_2} = (\alpha_{l_2})_{l_1}$ *and* $(\alpha^{l_1})^{l_2} = (\alpha^{l_2})^{l_1}$.

For the sake of Condition 3(3), we denote

$$\alpha_{l_1 l_2} := \left(\alpha_{l_1}\right)_{l_2} = \left(\alpha_{l_2}\right)_{l_1} \text{ and } \alpha^{l_1 l_2} := \left(\alpha^{l_1}\right)^{l_2} = \left(\alpha^{l_2}\right)^{l_1} \text{ for } \alpha \in \mathcal{P} \text{ and } l_1, l_2 \in \Phi \cup \{\#\}.$$

3.2 Preliminary lemma

For any $\alpha \in \mathcal{P}$, we denote:

$$\min r_\Phi \alpha := \min\left\{ r_l \alpha \,\middle|\, l \in \Phi \right\} \text{ and } \min r^\Phi \alpha := \min\left\{ r^l \alpha \,\middle|\, l \in \Phi \right\}.$$

Lemma 3. *The following hold for any $\alpha \in \mathcal{P}$ and $l, l_1, l_2 \in \Phi \cup \{\#\}$:*

(1) $r_l \alpha \geq r_\# \alpha$. *Eespecially* $\min r_\Phi \alpha \geq r_\# \alpha$.

(2) $r_l \alpha = r_\# \alpha$ *if and only if* $\alpha_l = \alpha_\#$.

(3) $r_l \alpha^\# \geq r^\# \alpha$, *Especially* $\min r_\Phi \alpha^\# \geq r^\# \alpha$.

(4) $r_l \alpha^\# = r^\# \alpha$ *if and only if* $\left(\alpha^\#\right)_l = \alpha$. *Especially in this case, we have* $\left(\alpha^\#\right)_l = \left(\alpha^\#\right)_\# = \alpha$.

(5) $\left(\alpha^{l_1}\right)_{l_1 l_2} \leq \alpha_{l_2}$, $\left(\alpha^{l_1 l_2}\right)_{l_1} \leq \alpha^{l_2}$, $\alpha^{l_2} \leq \left(\alpha_{l_1}\right)^{l_1 l_2}$ *and* $\alpha_{l_2} \leq \left(\alpha_{l_1 l_2}\right)^{l_1}$.

(6) *If* $r_l \alpha^\# = r^\# \alpha$, *then* $\left(\alpha^\#\right)_\# = \alpha$ *and* $r_l \alpha^\# = r_\# \alpha^\#$.

(7) *If* $\alpha \in \mathcal{P}^\#$ *and* $r_l \alpha = r_\# \alpha$, *then* $r_l \left(\alpha_\#\right)^\# = r^\# \alpha_\#$.

(8) $\min r_\Phi \alpha^\# \leq \underline{r}^\# \alpha$ *if and only if* $\min r_\Phi \alpha^\# \leq \underline{r}_\# \alpha^\#$.

Proof. (1): Since $\alpha_\# \geq \alpha_l$, we have $r_l \alpha - r_\# \alpha = \left(r\alpha - r\alpha_l\right) - \left(r\alpha - r\alpha_\#\right) = r\alpha_\# - r\alpha_l \geq 0$.

(2): $r_l \alpha = r_\# \alpha$ is equivalent to $r\alpha_l = r\alpha_\#$. So the condition $\alpha_l \leq \alpha_\#$ implies $\alpha_l = \alpha_\#$ and vice versa.

(3): We note $\left(\alpha^\#\right)_l \leq \left(\alpha^\#\right)_\# \leq \alpha$. Hence we have $r_l \alpha^\# - r^\# \alpha = \left(r\alpha^\# - r\left(\alpha^\#\right)_l\right) - \left(r\alpha^\# - r\alpha\right) = r\alpha - r\left(\alpha^\#\right)_l \geq 0$.

(4): $r_l \alpha^\# = r^\# \alpha$ is equivalent to $r\left(\alpha^\#\right)_l = r\alpha$. So the condition $\left(\alpha^\#\right)_l \leq \left(\alpha^\#\right)_\# \leq \alpha$ implies $\left(\alpha^\#\right)_l = \alpha$ and vice versa.

(5): Since $\left(\alpha^{l_1}\right)_{l_1} \leq \alpha$ we have $\left(\alpha^{l_1}\right)_{l_1 l_2} \leq \alpha_{l_2}$. From Condition 3(3), we have $\left(\alpha^{l_1 l_2}\right)_{l_1} = \left(\left(\alpha^{l_2}\right)^{l_1}\right)_{l_1} \leq \alpha^{l_2}$. The other inequalities can be easily verified by similar way.

(6): We note that $\alpha = \left(\alpha^\#\right)_\#$ by (4). Therefore $r_l \alpha^\# = r^\# \alpha = r^\# \left(\alpha^\#\right)_\# = r_\# \left(\left(\alpha^\#\right)_\#\right)^\# = r_\# \alpha^\#$.

(7): By Lemma 1(1),(3), $\alpha \in \mathcal{P}^\#$ implies $\left(\alpha_\#\right)^\# = \alpha$ and $r_\# \alpha = r_\# \left(\alpha_\#\right)^\# = r^\# \left(\left(\alpha_\#\right)^\#\right)_\# = r^\# \alpha_\#$. Therefore, from our assumption, we have $r_l \left(\alpha_\#\right)^\# = r_l \alpha = r_\# \alpha = r^\# \alpha_\#$.

(8) :This follows form $\underline{r}^\# \alpha = \underline{r}_\# \alpha^\#$ by Lemma 1(5). □

3.3 The m-full subset, the restrictedly full subset and their correspondence

Definition 3. *We define five subsets* $\mathfrak{m}\text{-Full}_\Phi \mathcal{P}$, $\text{w-}\mathfrak{m}\text{-Full}_\Phi \mathcal{P}$, $\text{Full}_\Phi \mathcal{P}$, $\text{w-Full}_\Phi \mathcal{P}$, $\text{Full}'_\Phi \mathcal{P}$ *as follows:*

$$\mathfrak{m}\text{-Full}_\Phi \mathcal{P} := \left\{ \alpha \in \mathcal{P} \,\middle|\, r_l \alpha^\# = r^\# \alpha \ \left(i.e. \left(\alpha^\#\right)_l = \alpha\right) \ for\ some\ l \in \Phi \right\} = \left\{ \alpha \in \mathcal{P} \,\middle|\, \min r_\Phi \alpha^\# = r^\# \alpha \right\},$$

$$\text{w-}\mathfrak{m}\text{-Full}_\Phi \mathcal{P} := \left\{ \alpha \in \mathcal{P} \,\middle|\, r_l \alpha^\# \leq \underline{r}^\# \alpha \ \ for\ some\ l \in \Phi \right\} = \left\{ \alpha \in \mathcal{P} \,\middle|\, \min r_\Phi \alpha^\# \leq \underline{r}^\# \alpha \right\},$$

$$\text{Full}_\Phi \mathcal{P} := \left\{ \alpha \in \mathcal{P} \,\middle|\, r_l \alpha = r_\# \alpha \ \left(i.e.\ \alpha_\# = \alpha_l\right) \ for\ some\ l \in \Phi \right\} = \left\{ \alpha \in \mathcal{P} \,\middle|\, \min r_\Phi \alpha = r_\# \alpha \right\},$$

$$\text{w-Full}_\Phi \mathcal{P} := \left\{ \alpha \in \mathcal{P} \,\middle|\, r_l \alpha \leq \underline{r}_\# \alpha \ \ for\ some\ l \in \Phi \right\} = \left\{ \alpha \in \mathcal{P} \,\middle|\, \min r_\Phi \alpha \leq \underline{r}_\# \alpha \right\},$$

$$\text{Full}'_\Phi \mathcal{P} := \text{Full}_\Phi \mathcal{P} \cap \mathcal{P}^\#,$$

and we call each of them "m-full subset", "weakly m-full subset", "full subset","weakly full subset" and "restrictedly full subset" of P w.r.t. Φ.

Remark 3. We *remark* $\mathfrak{m}\text{-Full}_\Phi \mathcal{P} \subseteq \text{w-}\mathfrak{m}\text{-Full}_\Phi \mathcal{P}$ *and* $\text{Full}_\Phi \mathcal{P} \subseteq \text{w-Full}_\Phi \mathcal{P}$.

Lemma 4. $\alpha \in \text{w-}\mathfrak{m}\text{-Full}_\Phi \mathcal{P}$ *if and only if* $\alpha^\# \in \text{w-Full}_\Phi \mathcal{P}$.

Proof. If $\alpha \in \text{w-}\mathfrak{m}\text{-Full}_\Phi \mathcal{P}$, then $r_l \alpha^\# \leq \underline{r}^\# \alpha$ for some $l \in \Phi$. Lemma 1(5) $\underline{r}^\# \alpha = \underline{r}_\# \alpha^\#$ implies $\alpha^\# \in \text{w-Full}_\Phi \mathcal{P}$. Conversely if $\alpha^\# \in \text{w-Full}_\Phi \mathcal{P}$, then $r_l \alpha^\# \leq \underline{r}_\# \alpha^\#$ for some $l \in \Phi$. Again by Lemma 1(5) $\underline{r}^\# \alpha = \underline{r}_\# \alpha^\#$, we have $\alpha \in \text{w-}\mathfrak{m}\text{-Full}_\Phi \mathcal{P}$. □

We can establish a one-to-one correspondence between $\mathfrak{m}\text{-Full}_\Phi \mathcal{P}$ and $\text{Full}'_\Phi \mathcal{P}$. We state this bellow as a theorem.

Theorem 2. *There are one-to-one correspondences:*

$$(1)\ \mathfrak{m}\text{-Full}_\Phi\mathcal{P} \underset{(\ )_\#}{\overset{(\ )^\#}{\rightleftarrows}} \text{Full}'_\Phi\mathcal{P}\ ,$$

$$(2)\ \text{w-}\mathfrak{m}\text{-Full}_\Phi\mathcal{P}\cap\mathcal{P}_\# \underset{(\ )_\#}{\overset{(\ )^\#}{\rightleftarrows}} \text{w-Full}_\Phi\mathcal{P}\cap\mathcal{P}^\#\ .$$

Proof. (1): follows immediately from Lemma 1(7),(8).

(2): If $\alpha\in\text{w-}\mathfrak{m}\text{-Full}_\Phi\mathcal{P}\cap\mathcal{P}_\#$, then $\alpha^\#\in\text{w-Full}_\Phi\mathcal{P}\cap\mathcal{P}^\#$ by Lemma 4 and $\left(\alpha^\#\right)_\#=\alpha$ by Lemma 1(1). Conversely if $\alpha\in\text{w-Full}_\Phi\mathcal{P}\cap\mathcal{P}^\#$, then $\left(\alpha_\#\right)^\#=\alpha$ by Lemma 1(1) and $\alpha_\#\in\text{w-}\mathfrak{m}\text{-Full}_\Phi\mathcal{P}\cap\mathcal{P}_\#$ by Lmma 4. □

3.4 The m-full closure and the full closure

For $\alpha\in\mathcal{P}$, if there exists an unique maximal element among those elements $\beta\le\alpha$ with $\beta\in\mathfrak{m}\text{-Full}_\Phi\mathcal{P}$, we call it "m-full closure of α" and denote it by $\tilde{\alpha}^{\mathfrak{m}\text{-}f}$. Similarly, if there exists an unique maximal element among those elements $\beta\le\alpha$ with $\beta\in\text{Full}_\Phi\mathcal{P}$, we call it "full closure of α" and denote it by $\tilde{\alpha}^{f}$. For any subset $\Omega\subseteq\mathcal{P}$, if there exists an unique minimal element among those elements $\beta\in\mathcal{P}$ such that $\beta\ge\omega$ for any $\omega\in\Omega$, we call it supremum of Ω and denote it by $\sup\Omega$.

Condition 4. *For any subset* $\Omega\subseteq\mathcal{P}$*, the supremum* $\sup\Omega$ *exists and the following condition are satisfied:*

$$\sup\Omega_l=\left(\sup\Omega\right)_l\ ,$$

where $l\in\Phi\cup\{\#\}$ *and* Ω_l *denotes the image of* Ω *by* $(\ )_l$.

Moreover we assume that the sets $\bigcap_{\alpha\in\mathfrak{m}\text{-Full}_\Phi\mathcal{P}}\left\{l\in\Phi\middle|\left(\alpha^\#\right)_l=\alpha\right\}$ *and* $\bigcap_{\alpha\in\text{Full}_\Phi\mathcal{P}}\left\{l\in\Phi\middle|\alpha_l=\alpha_\#\right\}$ *are both non-empty.*

Remark 4. *By the definition of supremum, it is always true that* $\sup\Omega^\#\le\left(\sup\Omega\right)^\#$.

Proposition 3. *If Condition 4 holds, then, for any* $\alpha\in\mathcal{P}$*, we have*

(1) $\tilde{\alpha}^{\mathfrak{m}\text{-}f}=\sup\left\{\beta\in\mathcal{P}\middle|\beta\le\alpha\ \textit{and}\ \beta\in\mathfrak{m}\text{-Full}_\Phi\mathcal{P}\right\}$,

(2) $\tilde{\alpha}^{f}=\sup\left\{\beta\in\mathcal{P}\middle|\beta\le\alpha\ \textit{and}\ \beta\in\text{Full}_\Phi\mathcal{P}\right\}$.

Proof. (1): We put $\gamma=\sup\left\{\beta\in\mathcal{P}\middle|\beta\le\alpha\ \textit{and}\ \beta\in\mathfrak{m}\text{-Full}_\Phi\mathcal{P}\right\}$. Taking an element $m\in\bigcap_{\alpha\in\mathfrak{m}\text{-Full}_\Phi\mathcal{P}}\left\{l\in\Phi\middle|\left(\alpha^\#\right)_l=\alpha\right\}$, we have

$$\left(\gamma^\#\right)_m=\sup\left\{\left(\beta^\#\right)_m\in\mathcal{P}\middle|\beta\le\alpha\ \textit{and}\ \beta\in\mathfrak{m}\text{-Full}_\Phi\mathcal{P}\right\}=\sup\left\{\beta\in\mathcal{P}\middle|\beta\le\alpha\ \textit{and}\ \beta\in\mathfrak{m}\text{-Full}_\Phi\mathcal{P}\right\}=\gamma\ .$$

Therefore $\gamma\in\mathfrak{m}\text{-Full}_\Phi\mathcal{P}$. Hence $\tilde{\alpha}^{\mathfrak{m}\text{-}f}=\gamma$.

(2): Similarly, we put $\delta=\sup\left\{\beta\in\mathcal{P}\middle|\beta\le\alpha\ \textit{and}\ \beta\in\text{Full}_\Phi\mathcal{P}\right\}$. Taking an element $n\in\bigcap_{\alpha\in\text{Full}_\Phi\mathcal{P}}\left\{l\in\Phi\middle|\alpha_l=\alpha_\#\right\}$, we have

$$\delta_n=\sup\left\{\beta_n\in\mathcal{P}\middle|\beta\le\alpha\ \textit{and}\ \beta\in\text{Full}_\Phi\mathcal{P}\right\}=\sup\left\{\beta_\#\in\mathcal{P}\middle|\beta\le\alpha\ \textit{and}\ \beta\in\text{Full}_\Phi\mathcal{P}\right\}=\delta_\#\ .$$

Therefore $\delta\in\text{Full}_\Phi\mathcal{P}$. Hence $\tilde{\alpha}^{f}=\delta$. □

3.5 Behavior of these properties under an order reversing bijection

At the end of this section, we describe how these properties behave under an order reversing bijection. Let $\mathcal{P}^\vee$ be a poset having two families of order preserving maps, $(\ )_l,(\ )^l:\mathcal{P}^\vee\to\mathcal{P}^\vee\ \left(l\in\Phi\cup\{\#\}\right)$ indexed by the same set $\Phi\cup\{\#\}$ as it in the case of $\mathcal{P}$, which satisfy Condition 3 with replacing $\mathcal{P}$ by $\mathcal{P}^\vee$. Let $(\ )^\circ:\mathcal{P}\to\mathcal{P}^\vee$ be an order reversing bijection which satisfies the following conditions.

Condition 5. *For any* $\alpha\in\mathcal{P}$ *and* $l\in\Phi\cup\{\#\}$,

$$\left(\alpha^\circ\right)^l=\left(\alpha_l\right)^\circ\ \ \textit{and}\ \ \left(\alpha^\circ\right)_l=\left(\alpha^l\right)^\circ.$$

As the manner in 2.4, if we put $q\alpha^\circ:=-r\alpha$, then $\mathcal{P}^\vee$ becomes a ranked poset with the rank function q. We denote $q^l\beta:=q\beta^l-q\beta$, $q_l\beta:=q\beta-q_l\beta$ and $\min q_\Phi\beta:=\min\left\{q_l\beta\middle|l\in\Phi\right\}$ for all $\beta\in\mathcal{P}^\vee$.

Under these conditions, we have the following lemma and proposition.

Lemma 5. *The following hold for any* $\alpha\in\mathcal{P}$ *and* $l\in\Phi\cup\{\#\}$ *:*

(1) $q^l\alpha^\circ=r_l\alpha$, $q_l\alpha^\circ=r^l\alpha$.

(2) $\underline{q}^l\alpha^\circ=\overline{r}_l\alpha$, $\underline{q}_l\alpha^\circ=\overline{r}^l\alpha$.

(3) $\min q_{\Phi}\alpha^{\circ}=\min r^{\Phi}\alpha$.

Proof. Proofs of (1) and (2) are quite similar to those of Lemma 2. So we omit. (3) follows form (1). □

Proposition 4. *For any* $\alpha\in\mathcal{P}$ *, the following hold:*

(1) $\alpha^{\circ}\in\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}^{\vee}$ *if and only if* $(\alpha_{\#})^{l}=\alpha$ *for some* $l\in\Phi$.

(2) $\alpha^{\circ}\in\text{w-}\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}^{\vee}$ *if and only if* $\overline{r_{\#}}\alpha\geq\min r^{\Phi}\alpha_{\#}$.

(3) $\alpha^{\circ}\in\text{Full}_{\Phi}\mathcal{P}^{\vee}$ *if and only if* $\alpha^{\#}=\alpha^{l}$ *for some* $l\in\Phi$.

(4) $\alpha^{\circ}\in\text{w-Full}_{\Phi}\mathcal{P}^{\vee}$ *if and only if* $\overline{r^{\#}}\alpha\geq\min r^{\Phi}\alpha$.

Proof. (1):Since $\left(\left(\alpha^{\circ}\right)^{\#}\right)_{l}=\left(\left(\alpha_{\#}\right)^{l}\right)^{\circ}$, $\alpha^{\circ}\in\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}^{\vee}$ if and only if $(\alpha_{\#})^{l}=\alpha$ for some $l\in\Phi$.

(2): Since $\underline{q^{\#}}\alpha^{\circ}=\overline{r_{\#}}\alpha$ and $\min q_{\Phi}\left(\alpha^{\circ}\right)^{\#}=\min q_{\Phi}\left(\alpha_{\#}\right)^{\circ}=\min r^{\Phi}\alpha_{\#}$ by Lemma 5(2) and (3), $\alpha^{\circ}\in\text{w-}\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}^{\vee}$ if and only if $\overline{r_{\#}}\alpha\geq\min r^{\Phi}\alpha_{\#}$.

(3):Since $\left(\alpha^{\circ}\right)_{\#}=\left(\alpha^{\#}\right)^{\circ}$ and $\left(\alpha^{\circ}\right)_{l}=\left(\alpha^{l}\right)^{\circ}$, $\alpha^{\circ}\in\text{Full}_{\Phi}\mathcal{P}^{\vee}$ if and only if $\alpha^{\#}=\alpha^{l}$ for some $l\in\Phi$.

(4):Since $\underline{q_{\#}}\alpha^{\circ}=\overline{r^{\#}}\alpha$ by Lemma 5(2) and $\min q_{\Phi}\alpha^{\circ}=\min r^{\Phi}\alpha$, $\alpha^{\circ}\in\text{w-Full}_{\Phi}\mathcal{P}^{\vee}$ if and only if $\overline{r^{\#}}\alpha\geq\min r^{\Phi}\alpha$. □

4. Relations among Rees property and its related properties

4.1 Preliminary lemma

We assume that the partially ordered set $\mathcal{P}$ satisfies Condition 3 in 3.1.

Lemma 6. *The following hold:*

(1) *If* $\left(\alpha^{\#}\right)_{l}\geq\beta$ *in* $\mathcal{P}$ *for some* $l\in\Phi$ *, then* $r_{l}\alpha^{\#}\geq r^{\#}\beta$ *. Especially* $r_{l}\alpha^{\#}\geq\underline{r^{\#}}\left(\alpha^{\#}\right)_{l}$.

(2) *If* $\alpha_{1}\geq\alpha_{2}$ *in* $\mathcal{P}$ *and* $l\in\Phi$ *, then* $r_{l}\alpha_{1}\geq r_{l}\alpha_{2}$ *. Moreover if* $r_{l}\alpha_{1}=r_{l}\alpha_{2}$ *, then* $\alpha_{1}\wedge\left(\left(\alpha_{2}\right)_{l}\right)^{l}=\alpha_{2}$ *. Especially* $r_{l}:\mathcal{P}\to\mathbb{Z}$ *is an order preserving map for any* $l\in\Phi$.

(3) $\min r_{\Phi}\alpha\geq\underline{r_{\#}}\alpha$ *for any* $\alpha\in\mathcal{P}$.

(4) $\min r_{\Phi}\alpha^{\#}\geq\underline{r^{\#}}\alpha$ *for any* $\alpha\in\mathcal{P}$.

Proof. (1): Since we have $\alpha\geq\left(\alpha^{\#}\right)_{\#}\geq\left(\alpha^{\#}\right)_{l}\geq\beta$ and $\alpha^{\#}\geq\beta^{\#}$, we get $\alpha^{\#}\wedge\beta^{l}\geq\alpha^{\#}\wedge\beta^{\#}=\beta^{\#}$. Hence it follows that $r_{l}\alpha^{\#}=r\left(\alpha^{\#}\wedge\beta^{l}\right)-r\beta\geq r\beta^{\#}-r\beta=r^{\#}\beta$.

(2) :Put $\beta=\left(\alpha_{2}\right)_{l}$ and notice $\alpha_{1}\wedge\beta^{l}\geq\alpha_{2}\wedge\beta^{l}$. From Condition 3(2), we have

$$r_{l}\alpha_{1}-r_{l}\alpha_{2}=\left(r\left(\alpha_{1}\wedge\beta^{l}\right)-r\beta\right)-\left(r\left(\alpha_{2}\wedge\beta^{l}\right)-r\beta\right)=r\left(\alpha_{1}\wedge\beta^{l}\right)-r\left(\alpha_{2}\wedge\beta^{l}\right)\geq0 .$$

Since $\beta^{l}=\left(\left(\alpha_{2}\right)_{l}\right)^{l}\geq\alpha_{2}$, we have $\alpha_{2}\wedge\beta^{l}=\alpha_{2}$. If $r_{l}\alpha_{1}=r_{l}\alpha_{2}$, this implies $\alpha_{1}\wedge\beta^{l}=\alpha_{1}\wedge\left(\left(\alpha_{2}\right)_{l}\right)^{l}=\alpha_{2}\wedge\beta^{l}=\alpha_{2}$.

(3):For any $\alpha\geq\beta$ and any $l\in\Phi$, we have $r_{l}\alpha\geq r_{l}\beta\geq r_{\#}\beta$ by (2) and Lemma 3(1). Therefore $\min r_{\Phi}\alpha\geq\underline{r_{\#}}\alpha$.

(4):From (3) and Lemma 1(5), we have $\min r_{\Phi}\alpha^{\#}\geq\underline{r_{\#}}\alpha^{\#}=\underline{r^{\#}}\alpha$. □

Remark 5. *From Lemma 6(3),* $\min r_{\Phi}\alpha\geq\underline{r_{\#}}\alpha$ *always holds, so we have*

$$\text{w-Full}_{\Phi}\mathcal{P}=\left\{\alpha\in\mathcal{P}\,\middle|\,\min r_{\Phi}\alpha=\underline{r_{\#}}\alpha\right\}.$$

Also, from Lemma 6(4), $\min r_{\Phi}\alpha^{\#}\geq\underline{r^{\#}}\alpha$ *always holds, so we have.*

$$\text{w-}\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}=\left\{\alpha\in\mathcal{P}\,\middle|\,\min r_{\Phi}\alpha^{\#}=\underline{r^{\#}}\alpha\right\}.$$

4.2 Relations among Rees property and its related properties

We investigate the inclusion relations holding among four subsets $\text{Rees}\mathcal{P}$, $\text{SRees}\mathcal{P}$, $\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}$ and $\text{Full}_{\Phi}\mathcal{P}$.

Theorem 3. *The following hold:*

(1) $\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}\subseteq\text{Rees}\mathcal{P}$.

(2) $\text{Full}_{\Phi}\mathcal{P}\subseteq\text{SRees}\mathcal{P}$.

(3) $\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}\subseteq\text{Full}_{\Phi}\mathcal{P}$.

(4) *If* $\alpha\in\text{w-}\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}$ *, then the following two conditions are equivalent:*

a) $\alpha\in\text{Rees}\mathcal{P}$.

b) $\alpha\in\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}$.

(5) *If* $\alpha \in \text{w-Full}_{\Phi}\mathcal{P}$ *, then the following two conditions are equivalent:*

a) $\alpha \in \text{SRees}\mathcal{P}$.

b) $\alpha \in \text{Full}_{\Phi}\mathcal{P}$.

Proof. (1): If $\alpha \in \mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}$, then we note that $\left(\alpha^{\#}\right)_{l} = \alpha$ and $r^{\#}\alpha = r_{l}\alpha^{\#}$ holds for some $l \in \Phi$. Using Lemma 6(1), we have $r^{\#}\alpha = r_{l}\alpha^{\#} \geq r^{\#}\beta$ for any β with $\alpha = \left(\alpha^{\#}\right)_{l} \geq \beta$. Hence we get $\alpha \in \text{Rees}\mathcal{P}$.

(2):If $\alpha \in \text{Full}_{\Phi}\mathcal{P}$, then using Lemma 6(2) and Lemma 3(1), we have $r_{\#}\alpha = r_{l}\alpha \geq r_{l}\beta \geq r_{\#}\beta$ for any β with $\alpha \geq \beta$. Hence we get $\alpha \in \text{SRees}\mathcal{P}$.

(3):If $\alpha \in \mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}$, then $\alpha = \left(\alpha^{\#}\right)_{\#} = \left(\alpha^{\#}\right)_{l}$ from Lemma 3(4). Therefore we have $\alpha_{l} = \left(\left(\alpha^{\#}\right)_{\#}\right)_{l} = \left(\left(\alpha^{\#}\right)_{l}\right)_{\#} = \alpha_{\#}$ by Condition 3(3). Hence we get $\alpha \in \text{Full}_{\Phi}\mathcal{P}$.

(4): Since $\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P} \subseteq \text{Rees}\mathcal{P}$ by (1), it is enough to show that $\alpha \in \mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}$ if $\alpha \in \text{w-}\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P} \cap \text{Rees}\mathcal{P}$. Now we assume that $\alpha \in \text{w-}\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P} \cap \text{Rees}\mathcal{P}$, then we note that $r^{\#}\alpha = \underline{r}^{\#}\alpha \geq \min r_{\Phi}\alpha^{\#}$ hold from the definitions. Using Lemma 3(3), we have $r^{\#}\alpha = \underline{r}^{\#}\alpha \geq \min r_{\Phi}\alpha^{\#} \geq r^{\#}\alpha$. Therefore $\min r_{\Phi}\alpha^{\#} = r^{\#}\alpha$. This implies $\alpha \in \mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}$.

(5): Since $\text{Full}_{\Phi}\mathcal{P} \subseteq \text{SRees}\mathcal{P}$ by (2), it is enough to show that $\alpha \in \text{Full}_{\Phi}\mathcal{P}$ if $\alpha \in \text{w-Full}_{\Phi}\mathcal{P} \cap \text{SRees}\mathcal{P}$. Now we assume that $\alpha \in \text{w-Full}_{\Phi}\mathcal{P} \cap \text{SRees}\mathcal{P}$, then we note that $r_{\#}\alpha = \underline{r_{\#}}\alpha \geq \min r_{\Phi}\alpha$ hold from the definitions. Using Lemma 3(1), we have $r_{\#}\alpha = \underline{r_{\#}}\alpha \geq \min r_{\Phi}\alpha \geq r_{\#}\alpha$. Therefore $\min r_{\Phi}\alpha = r_{\#}\alpha$. This implies $\alpha \in \text{Full}_{\Phi}\mathcal{P}$. □

Corollary1. *The following hold:*

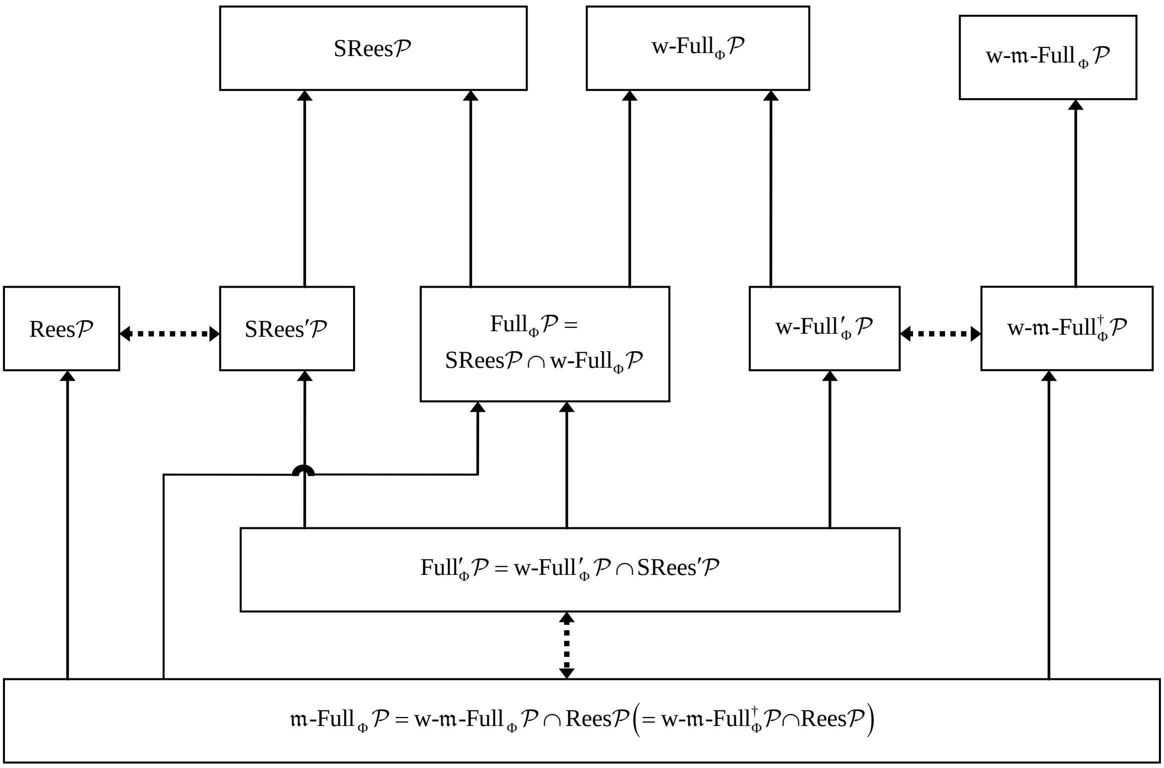
(1) $\alpha \in \text{Rees}\mathcal{P} \setminus \mathfrak{m}\text{-Full}_{\Phi}\mathcal{P}$ *if and only if* $\alpha \in \text{Rees}\mathcal{P}$ *and* $\underline{r}^{\#}\alpha < \min r_{\Phi}\alpha^{\#}$.

(2) $\alpha \in \text{SRees}\mathcal{P} \setminus \text{Full}_{\Phi}\mathcal{P}$ *if and only if* $\alpha \in \text{SRees}\mathcal{P}$ *and* $\underline{r_{\#}}\alpha < \min r_{\Phi}\alpha$.

5. Summary

We state the relations among Rees property and its related properties bellow as a diagram. In the daiagram bellow, a solid arrow A⟶B means that A implies B and a dotted arrow A◂⋯▸B means that there is a one-to-one correspondence between A and B. We denote:

$$\text{w-}\mathfrak{m}\text{-Full}^{\dagger}_{\Phi}\mathcal{P} := \text{w-}\mathfrak{m}\text{-Full}_{\Phi}\mathcal{P} \cap \mathcal{P}_{\#} \quad \text{w-Full}'_{\Phi}\mathcal{P} := \text{w-Full}_{\Phi}\mathcal{P} \cap \mathcal{P}^{\#} .$$



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