Paper

On q-anaolgue of Hypergeometoric Series

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Abstract In this paper, we treat a q-analogue of hypergeometric series. First, we define a q-analogue of hypergeometric series $f(x;\ \alpha,\ \beta,\ \lambda)$. And we find a q-hypergeometric differntial equation. Next, we consider local fundamental solutions at $x=0$ and $x=\infty$ in a q-hypergeometric differntial equation. Moreover, we obtain integral represents of each loca fundamental solution. Finally, we consider local fundamental solutions at $x=1$ in a q-hypergeometric differntial equation by using a change of variable.-

Keywords : Hypergeometric series, q-Hypergeometric differential equation, Local fundamental solutions, integral represent

1. A q-HYPERGEPMETORIC DIFFERENTIAL EQUATION

Let q be a fixed number with $0<q<1$. The hypergeometric series is defined by $f(x\ ;\ \alpha,\ \beta,\ \gamma)=\sum_{n=0}^{\infty}\frac{(\alpha,\ n)(\beta,\ n)}{(\gamma,\ n)(1,\ n)}x^n$,

where $(a,\ n)=\begin{cases}1 & (n=0)\\ a(a+1)(a+2)\cdots(a+n-1) & (n\geq 1)\end{cases}$ and $\alpha,\ \beta,\ \gamma$ are real constants

Then we define a q-analogue of hypergeometric series as

(1.1) $$f(x)=\sum_{n=0}^{\infty}\frac{(q^{\alpha}\ ;\ q)_n(q^{\beta}\ ;\ q)_n}{(q^{\gamma}\ ;\ q)_n(q\ ;\ q)_n}x^n$$

for $|x|<1$, where $(a\ ;\ q)_n=\begin{cases}1 & (n=0)\\ (1-a)(1-qa)\cdots(1-q^{n-1}a) & (n\geq 1)\end{cases}$.

Proposition 1. The function (1.1) is a solution of q-hypergeometoric differential equation

(1.2) $$x(1-x)\Delta_q\{\Delta_q f(x)\}+\{\gamma_q-(\alpha+\beta+1)_q x\}T_q\Delta_q f(x)-\alpha_q\beta_q T_q f(x)=0,$$

where $a_q=\frac{1-q^a}{1-q}$ is a q-analogue of number a , $T_q\varphi(x)=\varphi(qx)$ is a q-shift operator and $\Delta_q\varphi(x)=\frac{(1-T_q)\varphi(x)}{(1-q)x}$ is a q-differential operator.

Proof. When we put $a_n=\frac{(q^{\alpha}\ ;\ q)_n(q^{\beta}\ ;\ q)_n}{(q^{\gamma}\ ;\ q)_n(q\ ;\ q)_n}$, we have $\frac{a_{n+1}}{a_n}=\frac{(1-q^{\alpha+n})(1-q^{\beta+n})}{(1-q^{\gamma+n})(1-q^{n+1})}$.

Thus we can obtain an equation

(1.3) $$x\sum_{n=0}^{\infty}(\alpha+n)_q(\beta+n)_q a_n x^n=\sum_{n=0}^{\infty}(\gamma+n)_q(n+1)_q a_{n+1}x^{n+1}$$

$$=\sum_{n=0}^{\infty}(\gamma+n-1)_q n_q a_n x^n\ .$$

Now we define a differential operator D_{α} as $D_a\varphi(x)=x^{1-a}\Delta_q\{x^a\varphi(x)\}$. Then, from (1.3), we have

(1.4) $$xD_{\alpha}D_{\beta}f(x)=D_{\gamma-1}D_0 f(x)\ .$$

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Moreover, by directl calculation on D_α , we have

$$D_aD_b\varphi(x)=x^{1-a}\Delta_q[x^{1+a-b}\Delta_q\{x^b\varphi(x)\}]$$
$$=a_qb_q\varphi(x)+\{q^ab_q+q^b(a+1)_q\}x\Delta_q\varphi(x)+q^{a+b+1}x^2\Delta_q\{\Delta_q\varphi(x)\}$$
$$=a_qb_q\varphi(x)+\{(a+b+1)_q-(1-q)a_qb_q\}x\Delta_q\varphi(x)+\{1-(1-q)(a+b+1)_q\}x^2\Delta_q\{\Delta_q\varphi(x)\}$$
$$=a_qb_q\{1-(1-q)x\Delta_q\}\varphi(x)+(a+b+1)_qx\{1-(1-q)x\Delta_q\}\Delta_q\varphi(x)+x^2\Delta_q\{\Delta_q\varphi(x)\}$$
$$=a_qb_qT_q\varphi(x)+(a+b+1)_qxT_q\Delta_q\varphi(x)+x^2\Delta_q\{\Delta_q\varphi(x)\}\ .$$

Thus, from (1.4), we have

$$x(1-x)\Delta_q\{\Delta_qf(x)\}+\{\gamma_q-(\alpha+\beta+1)_qx\}T_q\Delta_qf(x)-\alpha_q\beta_qT_qf(x)=0$$ □

Remark. From $D_a\varphi(x)=\dfrac{(1-q^aT_q)\varphi(x)}{1-q}$, a solution of q-hypergeometoric differential equation (1.2) is equal to an equation

(1.5) $x(1-q^\alpha T_q)(1-q^\beta T_q)f(x)=(1-q^{\gamma-1}T_q)(1-T_q)f(x)$.

2. SEREIS EXPANSION

In this section, we find local solutions of q-hypergeometoric differential equation

(2.1) $x(1-q^\alpha T_q)(1-q^\beta T_q)\varphi(x)=(1-q^{\gamma-1}T_q)(1-T_q)\varphi(x)$.

Proposition 2. Local fundamental solutions of (2.1) at $x=0$ are

(2.2) $\varphi_1(x)=\sum_{n=0}^{\infty}\dfrac{(q^\alpha\ ;\ q)_n(q^\beta\ ;\ q)_n}{(q^\gamma\ ;\ q)_n(q\ ;\ q)_n}x^n$ （$|x|<1$）

and

(2.3) $\varphi_2(x)=x^{1-\gamma}\sum_{n=0}^{\infty}\dfrac{(q^{\alpha-\gamma+1}\ ;\ q)_n(q^{\beta-\gamma+1}\ ;\ q)_n}{(q^{2-\gamma}\ ;\ q)_n(q\ ;\ q)_n}x^n$ （$|x|<1$） .

Proof. We put $\varphi(x)=\sum_{n=0}^{\infty}a_nx^{n+\lambda}$ （$a_0\neq 0$） .

From (2.1), we have

$$\sum_{n=1}^{\infty}(1-q^{\alpha+n+\lambda-1})(1-q^{\beta+n+\lambda-1})a_{n-1}x^n=\sum_{n=0}^{\infty}(1-q^{\gamma+n+\lambda-1})(1-q^{n+\lambda})a_nx^n$$

Thus, we obtain

(2.4) $(1-q^{\gamma+\lambda-1})(1-q^\lambda)=0$

and

(2.5) $(1-q^{\alpha+n+\lambda-1})(1-q^{\beta+n+\lambda-1})a_{n-1}=(1-q^{\gamma+n+\lambda-1})(1-q^{n+\lambda})a_n\quad(n\geq 1)$.

From (2.4), we have $\lambda=0,\ 1-\gamma$. Thus, from (2.5), we obtain

$$a_n = \begin{cases} \dfrac{(q^{\alpha}\ ;\ q)_n (q^{\beta}\ ;\ q)_n}{(q^{\gamma}\ ;\ q)_n (q\ ;\ q)_n} a_0 & (\lambda = 0) \\ \dfrac{(q^{\alpha-\gamma+1}\ ;\ q)_n (q^{\beta-\gamma+1}\ ;\ q)_n}{(q^{2-\gamma}\ ;\ q)_n (q\ ;\ q)_n} a_0 & (\lambda = 1-\gamma) \end{cases} . \qquad \square$$

Proposition 3. Local fundamental solutions of (2.1) at $x = \infty$ are

(2.6) $$\varphi_3(x) = x^{-\alpha} \sum_{n=0}^{\infty} \frac{(q^{\alpha}\ ;\ q)_n (q^{\alpha-\gamma+1}\ ;\ q)_n}{(q^{\alpha-\beta+1}\ ;\ q)_n (q\ ;\ q)_n} q^{(\gamma-\alpha-\beta+1)n} x^{-n} \quad (|x| > q^{\gamma-\alpha-\beta+1})$$

and

(2.7) $$\varphi_4(x) = x^{-\beta} \sum_{n=0}^{\infty} \frac{(q^{\beta}\ ;\ q)_n (q^{\beta-\gamma+1}\ ;\ q)_n}{(q^{\beta-\alpha+1}\ ;\ q)_n (q\ ;\ q)_n} q^{(\gamma-\alpha-\beta+1)n} x^{-n} \quad (|x| > q^{\gamma-\alpha-\beta+1}) .$$

Proof. We put $\varphi(x) = \sum_{n=0}^{\infty} a_n x^{-n+\lambda}$ （$a_0 \neq 0$）.

From (2.1), we have

$$\sum_{n=0}^{\infty} (1-q^{\alpha-n+\lambda})(1-q^{\beta-n+\lambda}) a_n x^{-n} = \sum_{n=1}^{\infty} (1-q^{\gamma-n+\lambda})(1-q^{-n+\lambda+1}) a_{n-1} x^{-n}$$

Thus, we obtain

(2.8) $$(1-q^{\alpha+\lambda})(1-q^{\beta+\lambda}) = 0$$

and

(2.9) $$(1-q^{\alpha-n+\lambda})(1-q^{\beta-n+\lambda}) a_n = (1-q^{\gamma-n+\lambda})(1-q^{-n+\lambda+1}) a_{n-1} .$$

From (2.8), we have $\lambda = -\alpha,\ -\beta$ Thus, from (2.5), we obtain

$$a_n = \begin{cases} \dfrac{(p^{\alpha}\ ;\ p)_n (p^{\alpha-\gamma+1}\ ;\ p)_n}{(p^{\alpha-\beta+1}\ ;\ p)_n (p\ ;\ p)_n} a_0 & (\lambda = -\alpha) \\ \dfrac{(p^{\beta}\ ;\ p)_n (p^{\beta-\gamma+1}\ ;\ p)_n}{(p^{\beta-\alpha+1}\ ;\ p)_n (p\ ;\ p)_n} a_0 & (\lambda = -\beta) \end{cases} ,$$

where $p = q^{-1}$ and $(a\ ;\ p)_n = (1-a)(1-pa)\cdots(1-p^{n-1}a)$.

Moreover, by $(a\ ;\ p)_n = (-1)^n p^{\frac{1}{2}n(n-1)} a^n (a^{-1}\ ;\ q)_n$, we have

$$a_n = \begin{cases} \dfrac{(q^{\alpha}\ ;\ q)_n (q^{\alpha-\gamma+1}\ ;\ q)_n}{(q^{\alpha-\beta+1}\ ;\ q)_n (q\ ;\ q)_n} q^{(\gamma-\alpha-\beta+1)n} a_0 & (\lambda = -\alpha) \\ \dfrac{(q^{\beta}\ ;\ q)_n (q^{\beta-\gamma+1}\ ;\ q)_n}{(q^{\beta-\alpha+1}\ ;\ q)_n (q\ ;\ q)_n} q^{(\gamma-\alpha-\beta+1)n} a_0 & (\lambda = -\beta) \end{cases} . \qquad \square$$

3. INTEGRAL REPRESENTATION

In this section, we state each integral representation of local solutions of q-hypergeometoric differential equation (2.1).

We use the following notations;

- $[x]_q^n = x(qx)\cdots(q^{n-1}x) = q^{\frac{1}{2}n(n-1)} x^n$,
- $[x]_p^n = x(px)\cdots(p^{n-1}x) = p^{\frac{1}{2}n(n-1)} x^n$,
- $\begin{pmatrix} a \\ n \end{pmatrix}_q = \dfrac{(q^a\ ;\ p)_n}{(q\ ;\ q)_n}$,
- $\begin{pmatrix} a \\ n \end{pmatrix}_p = \dfrac{(p^a\ ;\ q)_n}{(p\ ;\ p)_n}$,

• $(x+[y]_q)^a = \sum_{n=0}^{\infty} \binom{a}{n}_q x^{a-n} [y]_q^n$, • $(x+[y]_p)^a = \sum_{n=0}^{\infty} \binom{a}{n}_p x^{a-n} [y]_p^n$,

• $\int_0^a \varphi(x)\ d_q x = \sum_{n=0}^{\infty} (1-q) q^n a\ \varphi(q^n a)$, • $\int_a^{\infty} \varphi(x)\ d_p x = -\sum_{n=0}^{\infty} (1-p) p^n a\ \varphi(p^n a)$.

Remark. Using a change of variable $x = 1/t$, the equation $\int_0^a \varphi(x)\ d_q x = \int_{\frac{1}{a}}^{\infty} \frac{\varphi(1/t)}{pt^2} d_p t$ is satisfied.

By using the q-binominal theorem (see (1))

$$\sum_{n=0}^{\infty} \frac{(a\ ;\ q)_n}{(q\ ;\ q)_n} x^n = \frac{(ax\ ;\ q)_\infty}{(x\ ;\ q)_\infty} \quad (|x| < 1) ,$$

where $(a\ ;\ q)_\infty = \lim_{n\to\infty} (a\ ;\ q)_n$, we can obtain the following lemma.

Lemma. The following equations are satisfied.

(3.1) $(x+[y]_q)^a = \frac{(-y/x\ ;\ q)_\infty}{(-q^a y/x\ ;\ q)_\infty} x^a \quad (|y/x| < q^{-\alpha})$

(3.2) $(x+[y]_p)^a = \frac{(-q^{1-\alpha} y/x\ ;\ q)_\infty}{(-qy/x\ ;\ q)_\infty} x^a \quad (|y/x| < q^{-1})$

Proof. We have

$$(x+[y]_q)^a = x^a \sum_{n=0}^{\infty} \frac{[-q^a]_p^n (q^{-a}\ ;\ q)_n}{(q\ ;\ q)_n} [y/x]_q^n = x^a \sum_{n=0}^{\infty} \frac{(q^{-a}\ ;\ q)_n}{(q\ ;\ q)_n} (-q^a y/x)^n = \frac{(-y/x\ ;\ q)_\infty}{(-q^a y/x\ ;\ q)_\infty} x^a$$

and

$$(x+[y]_p)^a = x^a \sum_{n=0}^{\infty} \frac{(p^a\ ;\ q)_n}{[-p]_p^n (p^{-1}\ ;\ q)_n} [y/x]_p^n = x^a \sum_{n=0}^{\infty} \frac{(q^{-a}\ ;\ q)_n}{(q\ ;\ q)_n} (-qy/x)^n = \frac{(-q^{1-\alpha} y/x\ ;\ q)_\infty}{(-qy/x\ ;\ q)_\infty} x^a$$

□

Remark. From (3.1) and (3.2), we have $(x+[y]_p)^a = (x+[q^{1-a} y]_q)^a$.

Numbers a and b are positive. In (1), a q-analogue of β-function is defined by $B_q(a,\ b) = \int_0^1 x^{a-1} \frac{(qx\ ;\ q)_\infty}{(q^b x\ ;\ q)_\infty} d_q x$. Hence we can rewrite $B_q(a,\ b) = \int_0^1 x^{a-1} (1-[qx]_q)^{b-1} d_q x$. And we note the equation $B_q(a,\ b) = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)}$ is satisfied, where a q-analogue of Γ-function is defined by $\Gamma_q(a) = \frac{(q\ ;\ q)_\infty}{(q^a\ ;\ q)_\infty} (1-q)^{1-a}$.

Theorem. If $0 < \alpha < \gamma < 1$ and $0 < \beta < 1$, the following equations are satisfied.

(3.3) $\int_0^1 \frac{t^{\alpha-1} (1-[qt]_q)^{\gamma-\alpha-1}}{(1-[tx]_q)^\beta} d_q t = B_q(\alpha,\ \gamma-\alpha)\ \varphi_1(x) \quad (|x|<1)$.

(3.4) $\int_{1/x}^{\infty} \frac{t^{\alpha-1} (qt-[1]_p)^{\gamma-\alpha-1}}{(tx-[1]_p)^\beta} d_p t = q^{\gamma-\alpha-2} B_q(\beta-\gamma+1,\ 1-\beta)\ \varphi_2(x) \quad (|x|<1)$.

(3.5) $\int_0^{q^{1-\beta}/x} \frac{t^{\alpha-1}(1-[qt]_q)^{\gamma-\alpha-1}}{(1-[xt]_q)^{\beta}} d_q t = q^{\alpha(1-\beta)} B_q(\alpha,\ 1-\beta)\varphi_3(x) \quad (|x| > q^{\gamma-\alpha-\beta+1})$.

(3.6) $\int_{q^{\alpha-\gamma}}^{\infty} \frac{t^{\alpha-1}(qt-[1]_p)^{\gamma-\alpha-1}}{(tx-[1]_p)^{\beta}} d_p t = q^{(\gamma-\alpha)(\beta-\gamma+2)-2} B_q(\beta-\gamma+1,\ \gamma-\alpha)\varphi_4(x) \quad (|x| > q^{\gamma-\alpha-\beta+1})$,

where $\varphi_1(x)$ and $\varphi_2(x)$ are defined by proposition 2, $\varphi_3(x)$ and $\varphi_4(x)$ are defined by proposition 3.

Proof. We have

$$\begin{aligned}
\varphi_1(x) &= \sum_{n=0}^{\infty} \frac{(q^{\alpha}\ ;\ q)_n (q^{\beta}\ ;\ q)_n}{(q^{\gamma}\ ;\ q)_n (q\ ;\ q)_n} x^n \\
&= \frac{(q^{\alpha}\ ;\ q)_{\infty}}{(q^{\gamma}\ ;\ q)_{\infty}} \sum_{n=0}^{\infty} \frac{(q^{\gamma+n}\ ;\ q)_{\infty} (q^{\beta}\ ;\ q)_n}{(q^{\alpha+n}\ ;\ q)_{\infty} (q\ ;\ q)_n} x^n \\
&= \frac{(q^{\alpha}\ ;\ q)_{\infty}}{(q^{\gamma}\ ;\ q)_{\infty}} \sum_{n=0}^{\infty} \frac{(q^{\beta}\ ;\ q)_n}{(q\ ;\ q)_n} x^n \sum_{k=0}^{\infty} \frac{(q^{\gamma-\alpha}\ ;\ q)_k}{(q\ ;\ q)_k} q^{(\alpha+n)k} \\
&= \frac{(q^{\alpha}\ ;\ q)_{\infty}}{(q^{\gamma}\ ;\ q)_{\infty}} \sum_{k=0}^{\infty} \frac{(q^{\gamma-\alpha}\ ;\ q)_k}{(q\ ;\ q)_k} q^{\alpha k} \sum_{n=0}^{\infty} \frac{(q^{\beta}\ ;\ q)_n}{(q\ ;\ q)_n} (q^k x)^n \\
&= \frac{(q^{\alpha}\ ;\ q)_{\infty} (q^{\gamma-\alpha}\ ;\ q)_{\infty}}{(q^{\gamma}\ ;\ q)_{\infty} (q\ ;\ q)_{\infty}} \sum_{k=0}^{\infty} \frac{(q^{k+1}\ ;\ q)_{\infty} (q^{\beta+k}x\ ;\ q)_{\infty}}{(q^{\gamma-\alpha+k}\ ;\ q)_{\infty} (q^k x\ ;\ q)_{\infty}} q^{\alpha k} \\
&= \frac{1}{B_q(\alpha,\ \gamma-\alpha)} \sum_{k=0}^{\infty} (1-q) q^k \cdot q^{k(\alpha-1)} \frac{(1-[q^{k+1}]_q)^{\gamma-\alpha-1}}{(1-[q^k x]_q)^{\beta}} \\
&= \frac{1}{B_q(\alpha,\ \gamma-\alpha)} \int_0^1 \frac{t^{\alpha-1}(1-[qt]_q)^{\gamma-\alpha-1}}{(1-[xt]_q)^{\beta}} d_q t\ .
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\varphi_2(x) &= \frac{1}{B_q(\beta-\gamma+1,\ 1-\beta)} \int_0^x \frac{t^{\beta-\gamma}(1-[t/q]_p)^{\gamma-\alpha-1}}{(x-[t]_p)^{\beta}} d_q t \\
&= \frac{1}{B_q(\beta-\gamma+1,\ 1-\beta)} \int_{1/x}^{\infty} \frac{t^{\gamma-\beta}(1-[1/tq]_p)^{\gamma-\alpha-1}}{pt^2(x-[1/t]_p)^{\beta}} d_p t \\
&= \frac{q^{\alpha-\gamma+2}}{B_q(\beta-\gamma+1,\ 1-\beta)} \int_{1/x}^{\infty} \frac{t^{\alpha-1}(qt-[1]_p)^{\gamma-\alpha-1}}{(xt-[1]_p)^{\beta}} d_p t\ ,
\end{aligned}$$

$$\varphi_3(x) = \frac{q^{\alpha(\beta-1)}}{B_q(\alpha,\ 1-\beta)} \int_0^{q^{1-\beta}/x} \frac{t^{\alpha-1}(1-[qt]_q)^{\gamma-\alpha-1}}{(1-[xt]_q)^{\beta}} d_q t$$

and

$$\begin{aligned}
\varphi_4(x) &= \frac{q^{(\alpha-\gamma)(\beta-\gamma+1)}}{B_q(\beta-\gamma+1,\ \gamma-\alpha)} \int_0^{q^{\gamma-\alpha}} \frac{t^{\beta-\gamma}(1-[q^{1+\alpha-\gamma}t]_q)^{\gamma-\alpha-1}}{(x-[q^{1-\beta}t]_q)^{\beta}} d_q t \\
&= \frac{q^{2+(\alpha-\gamma)(\beta-\gamma+2)}}{B_q(\beta-\gamma+1,\ \gamma-\alpha)} \int_{q^{\alpha-\gamma}}^{\infty} \frac{t^{\alpha-1}(qt-[1]_p)^{\gamma-\alpha-1}}{(tx-[1]_p)^{\beta}} d_p t
\end{aligned}$$

□

4. LOCAL FUNDAMENTAL SOLUTIONS AT X=1

In this section, we treat a change of variable $t = 1 - x$.

In detail, $q^n x$ correspons to $t=1-q^n x$. Thus, if $\varphi(q^n x)$ corresponds to $\psi(q^n t)=\varphi(1-q^n t)$. A q-shift operator $T_q\varphi(x)$ and a q-differential operator $\Delta_q\varphi(x)$ correspond to $T_q\psi(t)$ and $-\Delta_q\psi(t)$. Indeed, we have

$$T_q\varphi(x)=\varphi(qx) \ \to \ \varphi(1-qx)=\psi(qt)=T_q\psi(t)$$

and

$$\Delta_q\varphi(x)=\frac{\varphi(x)-\varphi(qx)}{x-qx} \ \to \ \frac{\varphi(1-t)-\varphi(1-qt)}{(1-t)-(1-qt)}=-\frac{\psi(t)-\psi(qt)}{t-qt}.$$

By this change of variable, we transform a solution of q-hypergeometoric differential equation

(4.1) $$x(1-x)\Delta_q\{\Delta_q\varphi(x)\}+\{\gamma_q-(\alpha+\beta+1)_q x\}T_q\Delta_q\varphi(x)-\alpha_q\beta_q T_q\varphi(x)=0.$$

Therefore, (4.1) corresponds to a q-differential equation

(4.2) $$t(1-t)\Delta_q\{\Delta_q\psi(t)\}+\{(\alpha+\beta+1)_q-\gamma_q-(\alpha+\beta+1)_q t\}T_q\Delta_q\psi(t)-\alpha_q\beta_q T_q\psi(t)=0.$$

Moreover, by substituting $\Delta_q\varphi(t)=\dfrac{(1-T_q)\varphi(t)}{(1-q)t}$ and $a_q=\dfrac{1-q^a}{1-q}$ for (4.2), we have

(4.3) $$t(1-q^\alpha T_q)(1-q^\beta T_q)\psi(t)=(1-T_q)\left\{1-(q^{-1}+q^{\alpha+\beta}-q^{\gamma-1})T_q\right\}\psi(t).$$

If $1+q^{\alpha+\beta+1}>q^\gamma$, we put $\delta=\log_q(1+q^{\alpha+\beta+1}-q^\gamma)$.

Then, from proposition 1, fundamental solution of (4.2) or (4.3) are

(4.4) $$\psi_5(t)=\sum_{n=0}^{\infty}\frac{(q^\alpha\ ;\ q)_n(q^\beta\ ;\ q)_n}{(q^\delta\ ;\ q)_n(q\ ;\ q)_n}t^n$$

and

(4.5) $$\psi_6(t)=t^{1-\delta}\sum_{n=0}^{\infty}\frac{(q^{\alpha-\delta+1}\ ;\ q)_n(q^{\beta-\delta+1}\ ;\ q)_n}{(q^{2-\delta}\ ;\ q)_n(q\ ;\ q)_n}t^n.$$

Therefore, as $q\to 1$, we have

(4.6) $$\lim_{q\to 1}\psi_5(t)=\sum_{n=0}^{\infty}\frac{(\alpha,\ n)(\beta,\ n)}{(\alpha+\beta-\gamma+1,\ n)(1,\ n)}t^n$$

and

(4.7) $$\lim_{q\to 1}\psi_6(t)=t^{\gamma-\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\gamma-\alpha,\ n)(\gamma-\beta,\ n)}{(\gamma-\alpha-\beta+1,\ n)(1,\ n)}t^n.$$

Remark. In (2), it is shown that local solutions of a q-differential equation $(1-x)\Delta_q f(x)+\alpha_q f(x)=0$ are $f(x)=(1-[x]_q)^\alpha$ at $x=0$ and $f(x)=(x-[1]_p)^\alpha$ at $x=\infty$. By using a change of variable $t=1-x$, this q-differential equation corresponds to $t\Delta_q g(t)+\alpha_q g(t)=0$, where $g(t)=f(1-t)$. And its fundamental solution is given by $g(t)=t^\alpha$.

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