Paper

On q-analogue of Difference Operator with Non-integer Order

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Abstract In this paper, we study a q-analogue of difference operator with non-integer order. The main theorem is that the composition rule for this operator is satisfied. And, by the result, we calculate the q-integral and q-differential with non-integer order for a q-exponential function.

Keywords : Riemann-Liouville fractional derivative, composition rule.

1. INTORODUCTION

In this paper, we study a q-analogue of difference operator with non-integer order.

For a function $f(x)$, the Rieman-Liouville fractional derivative ${}_aD_x^{-\alpha}$ is defined by

$$({}_aD_x^{-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\ dt\ . \tag{1.1}$$

In (1), for $a = 0$, $\alpha > 0$ and $x \neq 0$, a q-analogue of (1.1) is definded by

$$(\Delta_q^{-\alpha} f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-[qt])^{\alpha-1} f(t)\ d_qt\ . \tag{1.2}$$

In order to extend the range of α, we rewrite definition (1.2).

In the second section , we state a q-analogue of difference operator D_q^{α} with non-integr order and examples. In the third section, we state that the composition rule $D_q^{\alpha+\beta} = D_q^{\alpha} \circ D_q^{\beta}$ is satisfid. And, for proof of the composition rule, we treat a q-analogue of Pascal's traiangle, Leibniz formula and Vandermonde convolution formula. In the forth section, we calculate $D_q^{-\alpha} e_q(x),\ D_q^{1-\alpha} e_q(x)$ for $0 < \alpha < 1$.

Let q be a fixed number with $0 < q < 1$ and n be a natural number. We use the follwing notations:

- $(a\ ;\ q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a)$,
- $(a\ ;\ q)_0 = 1$ and $(a\ ;\ q)_\infty = \prod_{n=0}^{\infty}(1-q^n a)$,
- $n_q = 1 + q + \cdots + q^n = \frac{1-q^n}{1-q}$, q-natural number;
- $(n!)_q = 1_q \times 2_2 \times \cdots \times n_q = \frac{(q\ ;\ q)_n}{(1-q)^n}$, q-factorial;
- $\binom{n}{k}_q = \frac{(n!)_q}{(k!)_q\{(n-k)!\}_q} = \frac{(q^{n-k+1}\ ;\ q)_k}{(q\ ;\ q)_k}$ for $k = 0,\ 1,\ \cdots,\ n$, q-binomial coefficient;
- $\binom{\alpha}{k}_q = \frac{(q^{\alpha-k+1}\ ;\ q)_k}{(q\ ;\ q)_k}$, general q-binomial coefficient;

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- $(x-[y])^n=(x-y)(x-qy)\cdots(x-q^{n-1}y)=\dfrac{(y/x\,;\,q)_\infty}{(q^n y/x\,;\,q)_\infty}x^n$ for $x\neq 0$,

- $(x-[y])^\alpha=\dfrac{(y/x\,;\,q)_\infty}{(q^\alpha y/x\,;\,q)_\infty}x^\alpha$ for $|y/x|<q^{-\alpha}$.

2. DEFINITION

A q-analogue of difference operator with non-integer order is defined by

$$D_q^\alpha f(x)=\frac{1}{(1-q)^\alpha x^\alpha}\sum_{n=0}^{\infty}\frac{(q^{-\alpha}\,;\,q)_n}{(q\,;\,q)_n}\;q^n f(q^n x)\,. \tag{2.1}$$

Example 1. We have

(i) $D_q^1 f(x)=\dfrac{f(x)-f(qx)}{(1-q)x}=\Delta_q f(x)$, q-difference,

(ii) $D_q^{-1}f(x)=\sum_{n=0}^{\infty}(1-q)q^n x\; f(q^n x)=\int_0^x f(t)\;d_q t$, Jackson integral.

Example 2. By the q-binomial theorem $\dfrac{(ax\,;\,q)_\infty}{(x\,;\,q)_\infty}=\sum_{n=0}^{\infty}\dfrac{(a\,;\,q)_n}{(q\,;\,q)_n}x^n$ for $|x|<1$ in (2), we have

$$D_q^\alpha x^\lambda=\frac{x^{\lambda-\alpha}}{(1-q)^\alpha}\cdot\frac{(q^{1+\lambda-\alpha}\,;\,q)_\infty}{(q^{1+\lambda}\,;\,q)_\infty}\quad \text{for}\quad \lambda\neq -1\,. \tag{2.2}$$

Thus, for $n\in \mathrm{N}$, the following equations are satisfied:

(i) $D_q^n x^\lambda=(\lambda-n+1)_q\cdots(\lambda-1)_q\lambda_q x^{\lambda-n}$,

(ii) $D_q^{-n}x^\lambda=\dfrac{x^{\lambda+n}}{(\lambda+n)_q\cdots(\lambda+2)_q(\lambda+1)_q}$,

(iii) $D_q^\alpha x^n=\dfrac{x^{n-\alpha}}{\Gamma_q(1-\alpha)}\cdot\dfrac{(q\,;\,q)_n}{(q^{1-\alpha}\,;\,q)_n}$ $\quad(\alpha<1)$,

where $\Gamma_q(\lambda)$ is a q-analogue of Γ-function. It is defined by $\Gamma_q(\lambda)=\dfrac{(q\,;\,q)_\infty}{(q^\lambda\,;\,q)_\infty}\,(1-q)^{1-\lambda}$ for $\lambda>0$ in (2).

Example 3. For $n\in \mathrm{N}$, we have

$$\begin{aligned}
D_q^n f(x)&=\frac{1}{(1-q)^n x^n}\sum_{k=0}^{n}\frac{(q^{-n}\,;\,q)_k}{(q\,;\,q)_k}\;q^k f(q^k x)\\
&=\frac{1}{(1-q)^n x^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}_q q^{\frac{1}{2}k(k+1-2n)}f(q^k x)\\
&=\frac{1}{(1-q)^n x^n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{n-k}_q q^{\frac{1}{2}(n-k)(1-n-k)}f(q^{n-k}x)\\
&=\frac{1}{(1-q)^n q^{\frac{1}{2}n(n-1)}x^n}\sum_{k=0}^{n}\binom{n}{k}_q q^{\frac{1}{2}k(k-1)}(-T)^{n-k}f(x)\\
&=\frac{(1-T)(q-T)\cdots(q^{n-1}-T)f(x)}{(1-q)^n q^{\frac{1}{2}n(n-1)}x^n}\\
&=\Delta_q^n f(x)\,.
\end{aligned}$$

Example 4. For $\alpha > 0$ and $|x| > 0$, we have

$$D_q^{-\alpha} f(x) = (1-q)^{\alpha} x^{\alpha} \sum_{n=0}^{\infty} \frac{(q^a\ ;\ q)_n}{(q\ ;\ q)_n}\ q^n f(q^n x)$$

$$= \frac{1}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} (1-q) q^n x \cdot \frac{(q^{n+1}\ ;\ q)_{\infty}}{(q^{\alpha+n}\ ;\ q)_{\infty}} x^{\alpha-1} \cdot f(q^n x)$$

$$= \frac{1}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} (1-q) q^n x \cdot (x - [q^{n+1} x])^{\alpha-1} \cdot f(q^n x)$$

$$= \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-[qt])^{\alpha-1} f(t)\ d_q t\ .$$

Thus $D_q^{-\alpha} f(x)$ is a q-analogue of Riemann-Liouville integral（See(1)）.

3. MAIN THEOREM

Lemma 1（q-Pascal's triangle）.

Let n be a natural number. For $k = 1, \cdots, n$, we have

$$\binom{n+1}{k}_q = \binom{n}{k}_q + q^{n-k+1} \binom{n}{k-1}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q$$

Proof. We have

$$\binom{n}{k}_q + q^{n-k+1} \binom{n}{k-1}_q = \frac{(n!)_q}{(k!)_q \{(n-k)!\}_q} + \frac{q^{n-k+1} (n!)_q}{\{(k-1)!\}_q \{(n-k+1)!\}_q}$$

$$= \frac{(n!)_q}{(k!)_q \{(n+1-k)!\}_q} \left\{ (n+1-k)_q + q^{n+1-k} k_q \right\}$$

$$= \frac{\{(n+1)!\}_q}{(k!)_q \{(n+1-k)!\}_q} = \binom{n+1}{k}_q$$

Similarly, we can obtain $\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q$.

Lemma 2（q-Leibniz formula）.

We have

$$\Delta_q^n \{f(x) g(x)\} = \sum_{k=0}^{n} \binom{n}{k}_q\ \Delta_q^{n-k} f(x) \cdot T^{n-k} \Delta_q^k g(x) = \sum_{k=0}^{n} \binom{n}{k}_q\ T^k \Delta_q^{n-k} f(x) \cdot \Delta_q^k g(x)\ ,$$

where T is q-shift operator satisfied with $Tf(x) = f(qx)$.

Proof. Obviously,

$$\Delta_q \{f(x) g(x)\} = \Delta_q f(x) \cdot Tg(x) + f(x) \cdot \Delta_q g(x) = \Delta_q f(x) \cdot g(x) + Tf(x) \cdot \Delta_q g(x)$$

is satisfied.

By Lemma 1, we have

$$\Delta_q^{n+1} \{f(x) g(x)\} = \Delta_q \sum_{k=0}^{n} \binom{n}{k}_q\ \Delta_q^{n-k} f(x) \cdot T^{n-k} \Delta_q^k g(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k}_q\ \Delta_q^{n+1-k} f(x) \cdot T^{n+1-k} \Delta_q^k g(x) + \sum_{k=0}^{n} q^{n-k} \binom{n}{k}_q\ \Delta_q^{n-k} f(x) \cdot T^{n-k} \Delta_q^{k+1} g(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k}_q \Delta_q^{n+1-k} f(x) \cdot T^{n+1-k} \Delta_q^k g(x) + \sum_{k=1}^{n+1} q^{n+1-k} \binom{n}{k-1}_q \Delta_q^{n+1-k} f(x) \cdot T^{n+1-k} \Delta_q^k g(x)$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k}_q \Delta_q^{n+1-k} f(x) \cdot T^{n+1-k} \Delta_q^k g(x) .$$

Similarly, we can obtain

$$\Delta_q^{n+1}\{f(x)g(x)\} = \Delta_q \sum_{k=0}^{n} \binom{n}{k}_q T^k \Delta_q^{n-k} f(x) \cdot \Delta_q^k g(x)$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k}_q T^k \Delta_q^{n+1-k} f(x) \cdot \Delta_q^k g(x) .$$

Thus, by induction, we can obtain the result of Lemma 2.

Lemma 3（q-Vandermonde convolution formula）.

We have

$$\binom{\alpha+\beta}{n}_q = \sum_{k=0}^{n} \binom{\alpha}{n-k}_q \binom{\beta}{k}_q q^{(n-k)(\beta-k)} = \sum_{k=0}^{n} \binom{\alpha}{n-k}_q \binom{\beta}{k}_q q^{k(\alpha-n+k)}$$

Proof. We have the following equations;

- $\Delta_q^n x^{\alpha+B} = (n!)_q x^{\alpha+B-n} \binom{\alpha+\beta}{n}_q ,$
- $\sum_{k=0}^{n} \binom{n}{k}_q \Delta_q^{n-k} x^{\alpha} \cdot T^{n-k} \Delta_q^k x^{\beta} = (n!)_q x^{\alpha+\beta-n} \sum_{k=0}^{n} \binom{\alpha}{n-k}_q \binom{\beta}{k}_q q^{(n-k)(\beta-k)} ,$
- $\sum_{k=0}^{n} \binom{n}{k}_q T^k \Delta_q^{n-k} x^{\alpha} \cdot \Delta_q^k x^{\beta} = (n!)_q x^{\alpha+\beta-n} \sum_{k=0}^{n} \binom{\alpha}{n-k}_q \binom{\beta}{k}_q q^{k(\alpha-n+k)} .$

Thus, by Lemma 2, we can obtain the result of Lemma 3.

By relation $\frac{(q^{-\alpha}\,;\,q)_n}{(q\,;\,q)_n} q^{\alpha n} = (-1)^n q^{\frac{1}{2}n(n-1)} \binom{\alpha}{n}_q$, the result of Lemma 3 is also rewritten as follows:

$$\frac{(q^{-\alpha-\beta}\,;\,q)_n}{(q\,;\,q)_n} = \sum_{k=0}^{n} \frac{(q^{-\alpha}\,;\,q)_{n-k}}{(q\,;\,q)_{n-k}} \frac{(q^{-\beta}\,;\,q)_k}{(q\,;\,q)_k} q^{-\alpha k} = \sum_{k=0}^{n} \frac{(q^{-\alpha}\,;\,q)_{n-k}}{(q\,;\,q)_{n-k}} \frac{(q^{-\beta}\,;\,q)_k}{(q\,;\,q)_k} q^{-\beta(n-k)} . \tag{3.1}$$

Theorem. We have

$$D_q^{\alpha+\beta} f(x) = D_q^{\beta}\{D_q^{\alpha} f(x)\} .$$

Proof. By (3.1), we have

$$D_q^{\alpha+\beta} f(x) = \frac{1}{(1-q)^{\alpha+\beta} x^{\alpha+\beta}} \sum_{n=0}^{\infty} \frac{(q^{-\alpha-\beta}\,;\,q)_n}{(q\,;\,q)_n} q^n f(q^n x)$$

$$= \frac{1}{(1-q)^{\alpha+\beta} x^{\alpha+\beta}} \sum_{n=0}^{\infty} q^n f(q^n x) \sum_{k=0}^{n} \frac{(q^{-\alpha}\,;\,q)_{n-k}}{(q\,;\,q)_{n-k}} \cdot \frac{(q^{-\beta}\,;\,q)_k}{(q\,;\,q)_k} q^{-\alpha k}$$

$$= \frac{1}{(1-q)^{\alpha+\beta} x^{\alpha+\beta}} \sum_{k=0}^{\infty} \frac{(q^{-\beta}\,;\,q)_k}{(q\,;\,q)_k} q^{-\alpha k} \sum_{n=k}^{\infty} \frac{(q^{-\alpha}\,;\,q)_{n-k}}{(q\,;\,q)_{n-k}} q^n f(q^n x)$$

$$= \frac{1}{(1-q)^{\alpha+\beta} x^{\alpha+\beta}} \sum_{k=0}^{\infty} \frac{(q^{-\beta}\,;\,q)_k}{(q\,;\,q)_k} q^{-\alpha k} \sum_{n=0}^{\infty} \frac{(q^{-\alpha}\,;\,q)_n}{(q\,;\,q)_n} q^{n+k} f(q^{n+k} x)$$

$$= \frac{1}{(1-q)^{\beta} x^{\beta}} \sum_{k=0}^{\infty} \frac{(q^{-\beta}\,;\,q)_k}{(q\,;\,q)_k} q^k \cdot T^k \left[\frac{1}{(1-q)^{\alpha} x^{\alpha}} \sum_{n=0}^{\infty} \frac{(q^{-\alpha}\,;\,q)_n}{(q\,;\,q)_n} q^n f(q^n x) \right]$$

$$= D_q^{\beta}\{D_q^{\alpha} f(x)\}\ .$$

4. APPLICATIONS

In this section, we study the q-analogue of difference with non-integer on a exponential function.
In (2), two q-analogue of exponential function are definde by

$$e_q(x) = \frac{1}{((1-q)x\ ;\ q)_\infty} = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q\ ;\ q)_n} x^n \quad \text{for} \quad |x| < \infty_q = \lim_{n\to\infty} n_q = \frac{1}{1-q}$$

and

$$E_q(x) = (-(1-q)x\ ;\ q)_\infty \quad \text{for} \quad x \in \mathrm{R}$$

Lemma 4. For $|x| < \infty_q$, we have

$$\sum_{n=0}^{\infty} \frac{(1-q)^n}{(q^{\alpha+1}\ ;\ q)_n} x^{\alpha+n} = \alpha_q e_q(x) \int_0^x t^{\alpha-1} E_q(-qt)\ d_q t\ .$$

Proof. We put

$$f(x) = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q^{\alpha+1}\ ;\ q)_n} x^{\alpha+n}\ .$$

Then we have

$$\Delta_q f(x) = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q^{\alpha+1}\ ;\ q)_n} \cdot (\alpha+n)_q x^{\alpha+n-1}$$

$$= \alpha_q x^{\alpha-1} + \sum_{n=1}^{\infty} \frac{(1-q)^{n-1}}{(q^{\alpha+1}\ ;\ q)_{n-1}} x^{\alpha+n-1}$$

$$= \alpha_q x^{\alpha-1} + \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q^{\alpha+1}\ ;\ q)_n} x^{\alpha+n}$$

$$= \alpha_q x^{\alpha-1} + f(x)\ ,$$

and so

$$\Delta_q f(x) - f(x) = \alpha_q x^{\alpha-1}\ . \tag{4.1}$$

We use the variation of constant. We know that a solution of q-difference equation $\Delta_q f(x) - f(x) = 0$ is $f(x) = Ce_q(x)$. Thus we put $f(x) = C(x) \cdot e_q(x)$ and substitute it in (4.1). We have

$$\Delta_q C(x) \cdot e_q(qx) = \alpha_q x^{\alpha-1}\ ,$$

and so

$$C(x) = \alpha_q \int_0^x t^{\alpha-1} E_q(-qt)\ d_q t\ .$$

Hence we have

$$f(x) = \alpha_q e_q(x) \int_0^x t^{\alpha-1} E_q(-qt)\ d_q t\ .$$

Proposition 1. For $0 < \alpha < 1$, we have

$$D_q^{-\alpha} e_q(x) = e_q(x) \cdot Erf_q(x\ ;\ \alpha)\ , \tag{4.1}$$

where $Erf_q(x\ ;\ \alpha) = \frac{1}{\Gamma_q(\alpha)} \int_0^x t^{\alpha-1} E_q(-qt)\ d_q t\ .$

Proof. By Example 2(ⅲ) and Lemma 4, we have

$$D_q^{-\alpha}e_q(x)=\frac{1}{\Gamma_q(\alpha+1)}\sum_{n=0}^{\infty}\frac{(1-q)^n}{(q^{\alpha+1}\ ;\ q)_n}x^{\alpha+n}$$

$$=\frac{e_q(x)}{\Gamma_q(\alpha)}\int_0^x t^{\alpha-1}E_q(-qt)\ d_qt$$

$$=e_q(x)\cdot Erf_q(x\ ;\ \alpha)\ .$$

Proposition 2. For $0<\alpha<1$, we have

$$D_q^{1-\alpha}e_q(x)=\frac{1}{\Gamma_q(\alpha)x^{1-\alpha}}+e_q(x)\cdot Erf_q(x\ ;\ \alpha)\ . \tag{4.2}$$

Proof. By Theorem and Proposition 1, we have

$$D_q^{1-\alpha}e_q(x)=D_q^1\{D_q^{-\alpha}e_q(x)\}$$

$$=\Delta_q\left\{e_q(x)\cdot Erf_q(x\ ;\ \alpha)\right\}$$

$$=e_q(x)\cdot Erf_q(x\ ;\ \alpha)+e_q(qx)\cdot\frac{x^{\alpha-1}E_q(-qx)}{\Gamma_q(\alpha)}$$

$$=\frac{1}{\Gamma_q(\alpha)x^{1-\alpha}}+e_q(x)\cdot Erf_q(x\ ;\ \alpha)\ .$$

Remark. For $\alpha=\frac{1}{2}$ and $x>0$, we have the follwing equations:

- $\lim_{q\to1}D_q^{-\frac{1}{2}}e_q(x)=e^xErf(\sqrt{x})$,
- $\lim_{q\to\infty}D_q^{\frac{1}{2}}e_q(x)=\frac{1}{\sqrt{\pi x}}+e^xErf(\sqrt{x})$,

where $Erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\ dt$ is the error function.

Proof. We have

$$\lim_{q\to1}\Gamma_q\left(\frac{1}{2}\right)=\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

and

$$\lim_{q\to1}Erf_q\left(x\ ;\ \frac{1}{2}\right)=\frac{1}{\sqrt{\pi}}\int_0^x t^{-\frac{1}{2}}e^{-t}\ dt=\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}}e^{-t^2}\ dt\ .$$

Thus the result of remark is given from (4.1) and (4.2).

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