論文

On q-Laguerre Polynomials

Nobuo Kobachi*

Abstract In this paper, we study a q-analogue of Laguerre polynomials. We give a q-analogue of Rodrigues' formula, recurrence formulas, Laguerre equations and orthogonality.

Keywords : Laguerre polynomial, Rodrigues' formula, recurrence formulas, Laguerre equations, orthogonality

1. INTORODUCTION

In this paper, we study a q-analog of Laguerre polynomials. We give a q-analogue of Rodrigues' formula, recurrence formulas, Laguerre equations and orthogonality.

Let q be a fixed number with $0 < q < 1$.

We use the following notations:

- $n_q = 1 + q + \cdots + q^{n-1} = \dfrac{1-q^n}{1-q}$, q-natural number;
- $(n!)_q = 1_q \times 2_q \times \cdots \times n_q$, q- factorial;
- $\begin{pmatrix} n \\ k \end{pmatrix}_q = \dfrac{(n!)_q}{(k!)_q\{(n-k)!\}_q}$, q- binomial coefficient;
- $\infty_q = \lim\limits_{n\to\infty} n_q = \dfrac{1}{1-q}$, q-infinity ;
- $[a]^n = a\cdot(qa)\cdot\cdots\cdot(q^{n-1}a) = q^{\frac{1}{2}n(n-1)}a^n$, q-shift n-th power.;
- $Tf(x) = f(qx)$, q-shift operato;
- $\Delta_q f(x) = \dfrac{(1-T)f(x)}{(1-q)x}$, q-difference operator;
- $\int_0^a f(x)\ d_q x = \sum\limits_{n=0}^{\infty}(1-q)q^n af(q^n a)$, Jackson's integral.

See (1). Both

$$e_q(x) = \sum_{n=0}^{\infty}\frac{x^n}{(n!)_q} = \frac{1}{((1-q)x\ ;\ q)_\infty} \qquad (|x| < \infty_q)$$

and

$$E_q(x) = \sum_{n=0}^{\infty}\frac{[x]^n}{(n!)_q} = (-(1-q)x\ ;\ q)_\infty$$

mean q-analogues of exponential functions e^x, where

$$(a\ ;\ q)_\infty = \prod_{n=0}^{\infty}(1-q^n a).$$

For $0 \le x \le \infty_q$, a q-analogue of Laguerre polynomial is defined by

* Faculty of Liberal Studies, 2627 Hirayama-shinmachi, Yatsushiro-shi, Kumamoto Japan 866-8501

$$L_n(x) = \frac{1}{[-1]^n (n!)_q} \sum_{k=0}^{n} [-1]^k \binom{n}{k}_q \Delta_q^k (x^n) . \tag{1.1}$$

Indeed, for $n = 0,1,2,3$, we have

$$L_0(x) = 1 ,$$

$$L_1(x) = 1 - x ,$$

$$L_2(x) = \frac{1}{q \cdot (2!)_q} \left\{ q \cdot 2_q - 2_q^2 x + x^2 \right\} ,$$

$$L_3(x) = \frac{1}{q^3 (3!)_q} \left\{ q^3 (3!)_q - q \cdot 3_q^2 \cdot 2_q x + 3_q^2 x^2 - x^3 \right\} .$$

Then, in relation to Rodrigues' formula, recurrence formulas, Laguerre equations, orthogonality, we obtain the following theorem:

Theorem. We have
) Rodrigues' formula.

$$L_n(x) = e_q(qx) \Delta_q^n \left\{ E_q\left(-\frac{x}{q^{n-1}} \right) \frac{x^n}{(n!)_q} \right\} . \tag{1.2}$$

) Recurrence formulas.

$$(x\Delta_q - n_q) L_n(x) = -n_q L_{n-1}(x) , \tag{1.3}$$

$$L_n(qx) = L_{n-1}(x) - q \int_0^x L_{n-1}(t) \ d_q t . \tag{1.4}$$

) Laguerre equations.

$$q^n x \Delta_q^2 L_n(x) + (q^{n-1} - x) \Delta_q L_n(x) + n_q L_n(x) = 0 , \tag{1.5}$$

$$\Delta_q \left\{ E_q(-x) x \Delta_q L_n\left(\frac{x}{q} \right) \right\} = -\frac{n_q}{q^n} E(-qx) L_n(x) . \tag{1.6}$$

) Orthogonality.

$$\int_0^{\infty_q} E_q(-qx) L_n(x) L_m(x) \ d_q x = q^n \delta_{nm} , \tag{1.7}$$

where δ_{nm} is the Kroneker's delta.

These proofs are shown in each section after this.

2. RODRIGUES' FORMULA

In this section, we prove q-analogue of Rodrigues' formula.

We note that two functions $e_q(x)$, $E_q(x)$ have the following properties:

$$\Delta_q^n e_q(ax) = a^n e_q(ax) , \quad \Delta_q^n E_q(ax) = [a]^n E_q(q^n ax) .$$

By q-Leibniz formula (see Appendix), we have

$$\begin{aligned} \Delta_q^n \left\{ E_q\left(-\frac{x}{q^{n-1}} \right) \frac{x^n}{(n!)_q} \right\} &= \frac{1}{(n!)_q} \sum_{k=0}^{n} \binom{n}{k}_q T^k \left\{ \Delta_q^{n-k} E_q\left(-\frac{x}{q^{n-1}} \right) \right\} \cdot \Delta_q^k (x^n) \\ &= \frac{1}{[-1]^n (n!)_q} \sum_{k=0}^{n} \binom{n}{k}_q T^k \left\{ [-1]^k E_q\left(-\frac{x}{q^{k-1}} \right) \right\} \cdot \Delta_q^k (x^n) \end{aligned}$$

$$= \frac{E_q(-qx)}{[-1]^n} \sum_{k=0}^{n} [-1]^k \binom{n}{k}_q \Delta_q^k (x^n)$$

$$= E_q(-qx) L_n(x) .$$

Thus, by $e_q(x) \times E_q(-x) = 1$, we have

$$L_n(x) = e_q(qx) \Delta_q^n \left\{ E_q\left(-\frac{x}{q^{n-1}}\right) \frac{x^n}{(n!)_q} \right\},$$

and so we can obtain Rodrigues' formula.

3. RECURRENCE FORMULAS

In this section, we give two recurrence formulas.
We put

$$l_n(x) = \sum_{k=0}^{n} [-1]^k \binom{n}{k}_q \Delta_q^k (x^n)$$

$$= \sum_{k=0}^{n} [-1]^k \binom{n}{k}_q \binom{n}{k}_q (k!)_q x^{n-k} .$$

Then we have

$$x\Delta_q l_n(x) = \sum_{k=0}^{n} [-1]^k \binom{n}{k}_q \binom{n}{k}_q (k!)_q (n-k)_q x^{n-k}$$

$$= \sum_{k=0}^{n} [-1]^k \binom{n}{k}_q \binom{n}{k}_q (k!)_q (n_q - q^{n-k} k_q) x^{n-k}$$

$$= n_q l_n(x) + q^{n-1} \sum_{k=1}^{n} [-1]^{k-1} \binom{n}{k}_q \binom{n}{k}_q (k!)_q k_q x^{n-k} ,$$

and so

$$(x\Delta_q - n_q) l_n(x) = q^{n-1} n_q^2 \sum_{k=1}^{n} [-1]^{k-1} \binom{n-1}{k-1}_q \binom{n-1}{k-1}_q \{(k-1)!\}_q x^{n-k}$$

$$= q^{n-1} n_q^2 \sum_{k=0}^{n-1} [-1]^k \binom{n-1}{k}_q \binom{n-1}{k}_q (k!)_q x^{n-1-k}$$

$$= q^{n-1} n_q^2 l_{n-1}(x) .$$

Thus, by dividing $[-1]^n (n!)_q$, we have

$$(x\Delta_q - n_q) L_n(x) = -n_q L_{n-1}(x) .$$

On the other hand, by a q-analogue of Pascal Triangle Identity

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q ,$$

we have

$$l_n(qx) = \sum_{k=0}^{n} [-1]^k \binom{n}{k}_q \binom{n}{k}_q (k!)_q (qx)^{n-k}$$

$$= \sum_{k=1}^{n} [-1]^k \binom{n-1}{k-1}_q \binom{n}{k}_q (k!)_q (qx)^{n-k} + q^n \sum_{k=0}^{n-1} [-1]^k \binom{n-1}{k}_q \binom{n}{k}_q (k!)_q x^{n-k}$$

$$= -q^{n-1} n_q \sum_{k=0}^{n-1} [-1]^k \binom{n-1}{k}_q \binom{n-1}{k}_q (k!)_q \, x^{n-1-k} \; + q^n n_q \sum_{k=0}^{n-1} [-1]^k \binom{n-1}{k}_q \binom{n-1}{k}_q (k!)_q \frac{x^{n-k}}{n-k}$$

$$= -q^{n-1} n_q l_{n-1}(x) \; + q^n n_q \int_0^x l_{n-1}(t) \; d_q t \, .$$

Thus, by dividing $[-1]^n (n!)_q$, we have

$$L_n(qx) = L_{n-1}(x) - q \int_0^x L_{n-1}(t) \; d_q t \, .$$

4. LAGUERRE EQUATIONS

In this section, we give two q-analogues of Laguerre equation.

By (1.3) and (1.4), we have

$$n_q \Delta_q \{TL_n(x)\} = n_q \{\Delta_q L_{n-1}(x) - qL_{n-1}(x)\}$$

$$= (\Delta_q - q)\{n_q L_{n-1}(x)\}$$

$$= (\Delta_q - q)(n_q - x\Delta_q) L_n(x) \, ,$$

and so

$$qx\Delta_q^2 L_n(x) + (1-qx)\Delta_q L_n(x) + qn_q L_n(x) \; = n_q \Delta_q (1-T) L_n(x)$$

$$= n_q \Delta_q \{(1-q)x\Delta_q L_n(x)\}$$

$$= (1-q^n)\{\Delta_q L_n(x) + qx\Delta_q^2 L_n(x)\} \, .$$

Thus we have

$$q^n x\Delta_q^2 L_n(x) + (q^{n-1} - x)\Delta_q L_n(x) + n_q L_n(x) = 0 \, .$$

And furthermore, from this difference equation, we have

$$q^n x\Delta_q^2 L_n(x) + (q^{n-1} - q^n x)\Delta_q L_n(x) \; = (1-q^n) x\Delta_q L_n(x) - n_q L_n(x)$$

$$= -n_q \{1 - (1-q)x\Delta_q\} L_n(x)$$

$$= -n_q TL_n(x) \, ,$$

and so

$$x\Delta_q^2 L_n(x) + (q^{-1} - x)\Delta_q L_n(x) + q^{-n} n_q TL_n(x) = 0 \, .$$

Since $T^{-1}\Delta_q = q\Delta_q T^{-1}$, we have

$$qx\Delta_q^2 \left\{ L_n\left(\frac{x}{q}\right) \right\} + (1-x)\Delta_q \left\{ L_n\left(\frac{x}{q}\right) \right\} + q^{-n} n_q L_n(x) = 0 \, .$$

We note that

$$\Delta_q \left\{ E_q(-x) \cdot x \cdot \Delta_q f(x) \right\} \; = E_q(-qx)\{qx\Delta_q^2 f(x) + (1-x)\Delta_q f(x)\} \, .$$

Then we have

$$\Delta_q \left\{ E_q(-x) \cdot x \cdot \Delta_q L_n\left(\frac{x}{q}\right) \right\} \; = E_q(-qx) \left[qx\Delta_q^2 \left\{ L_n\left(\frac{x}{q}\right) \right\} + (1-x)\Delta_q \left\{ L_n\left(\frac{x}{q}\right) \right\} \right]$$

$$= -\frac{n_q}{q^n} E_q(-qx) L_n(x) \, .$$

5. ORTHOGONALITY

We put $\lambda_n = -q^{-n} n_q$ and

$$(f(x),\ g(x)) = \int_0^{\infty_q} E_q(-qx) f(x) g(x)\ d_q x .$$

First, we show that $(L_n(x),\ L_m(x)) = 0$ if $n \neq m$.

From (1.6) and $qT\Delta_q = \Delta_q T$, we have

$$\Delta_q \left\{ E_q(-x) x \Delta_q L_n\left(\frac{x}{q}\right) L_m(x) \right\} = \Delta_q \left\{ E_q(-x) x \Delta_q L_n\left(\frac{x}{q}\right) \right\} L_m(x) + T\left\{ E_q(-x) x \Delta_q L_n\left(\frac{x}{q}\right) \right\} \Delta_q L_m(x)$$

$$= \lambda_n E_q(-qx) L_n(x) L_m(x) + E_q(-qx) x \Delta_q L_n(x) \Delta_q L_m(x) .$$

Since $\int_0^a \Delta_q f(x)\ d_q x = f(a) - f(0)$, we have

$$\lambda_n (L_n(x),\ L_m(x)) = -\int_0^{\infty_q} E_q(-qx) x \Delta_q L_n(x) \Delta_q L_m(x)\ d_q x . \tag{5.1}$$

Similarly, we have

$$\lambda_m (L_m(x),\ L_n(x)) = -\int_0^{\infty_q} E_q(-qx) x \Delta_q L_m(x) \Delta_q L_n(x)\ d_q x . \tag{5.2}$$

Thus, by subtracting (5.1) from (5.2), we have

$$(\lambda_n - \lambda_m)(L_n(x),\ L_m(x)) = 0 .$$

Hence we obtain

$$(L_n(x),\ L_m(x)) = 0 \text{, if } n \neq m . \tag{5.3}$$

Next, we show $(L_n(x),\ L_n(x)) = q^n$.

For x^n, there exist constant numbers $c_0,\ c_1,\ \cdots,\ c_n$ such that

$$x^n = \sum_{k=0}^{n} c_k L_k(x) . \tag{5.4}$$

Then, by (5,3), the following properties satisfy

$$(x^k,\ L_n(x)) = 0 \quad (k = 0,\ 1,\ \cdots,\ n-1),$$

$$(x^n,\ L_n(x)) = [-1]^n (n!)_q (L_n(x),\ L_n(x))$$

Thus, by (5.1) and (1.3), we have

$$\lambda_n [-1]^n (n!)_q (L_n(x),\ L_n(x)) = \lambda_n (x^n,\ L_n(x))$$

$$= -n_q (x^{n-1},\ x\Delta_q L_n(x))$$

$$= -n_q^2 (x^{n-1},\ L_n(x) - L_{n-1}(x))$$

$$= n_q [-1]^{n-1} (n!)_q (L_{n-1}(x),\ L_{n-1}(x)) .$$

and so

$$(L_n(x),\ L_n(x)) = q(L_{n-1}(x),\ L_{n-1}(x)) .$$

Hence, from this recurrence formula, we have

$$(L_n(x),\ L_n(x)) = q^n (L_0(x),\ L_0(x))$$

$$= q^n \int_0^{\infty_q} E_q(-qx)\ d_q x$$

$$= q^n \left[-E_q(-x) \right]_0^{\infty_q} .$$

$$= q^n$$

Appendix

Lemma (q-Leibniz formula). We have

$$\Delta_q^n \{ f(x)g(x) \} = \sum_{k=0}^{n} \binom{n}{k}_q \{ T^k \Delta_q^{n-k} f(x) \} \Delta_q^k g(x) .$$

proof. Obviously, a q-Leibniz formula is satisfied for $n = 1$, so that

$$\Delta_q \{ f(x)g(x) \} = \Delta_q f(x) \cdot g(x) + Tf(x) \cdot \Delta_q g(x) .$$

And we have

$$\Delta_q^{n+1} \{ f(x)g(x) \} = \Delta_q \sum_{k=0}^{n} \binom{n}{k}_q \{ T^k \Delta_q^{n-k} f(x) \} \Delta_q^k g(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k}_q \{ q^k T^k \Delta_q^{n+1-k} f(x) \} \Delta_q^k g(x) + \sum_{k=0}^{n} \binom{n}{k}_q \{ T^{k+1} \Delta_q^{n-k} f(x) \} \Delta_q^{k+1} g(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k}_q \{ q^k T^k \Delta_q^{n+1-k} f(x) \} \Delta_q^k g(x) + \sum_{k=1}^{n+1} \binom{n}{k-1}_q \{ T^k \Delta_q^{n+1-k} f(x) \} \Delta_q^k g(x)$$

$$= \Delta^{n+1} f(x) \cdot g(x) + T^{n+1} f(x) \cdot \Delta_q^{n+1} g(x) + \sum_{k=1}^{n} \left\{ \binom{n}{k-1}_q + q^k \binom{n}{k}_q \right\} \{ T^k \Delta_q^{n+1-k} f(x) \} \Delta_q^k g(x)$$

$$= \Delta^{n+1} f(x) \cdot g(x) + T^{n+1} f(x) \cdot \Delta_q^{n+1} g(x) + \sum_{k=1}^{n} \binom{n+1}{k}_q \{ T^k \Delta_q^{n+1-k} f(x) \} \Delta_q^k g(x)$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k}_q \{ T^k \Delta_q^{n+1-k} f(x) \} \Delta_q^k g(x) .$$

Thus, by induction, a q-Leibniz formula is satisfied for all natural number n.

References

(1) G.E.Andrews, R.Askey and R.Roy : Special Functions, Cambridge Univ. Press, (1999)
(2) G.B.Folland : Fourier Analysis and Its Applications, American Math. Soc., (1992)
(3) 小野寺嘉孝 ： 物理のための応用数学, 掌華房, (1988)